

Stat312: Midterm Exam II.

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1. The amount of lateral expansion was determined for a sample of 15 pulsed-power gas metal arc weld used in LNG ship containment tanks. The measurements and necessary R output are given below. Assuming normality, derive a 99% confidence interval for population standard deviation σ [5pts].

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> mean(Y)
[1] 24.73333
> var(Y)
[1] 10.78095
> qchisq(c(0.005, 0.01, 0.995, 0.99), 15)
[1] 4.60 5.23 32.80 30.58
> qchisq(c(0.005, 0.01, 0.995, 0.99), 14)
[1] 4.07 4.66 31.32 29.14
```

Solution. $(n-1)S^2/\sigma^2 \sim \chi_{n-1}^2$. $\alpha = 0.01$. To construct a 99% CI for σ^2 , we need to compute critical values $\chi_{\alpha/2, n-1}^2 = \text{qchisq}(0.005, 14) = 4.07$ and $\chi_{1-\alpha/2, n-1}^2 = \text{qchisq}(0.995, 14) = 31.32$. Then 99% CI for σ^2 is

$$\left(\frac{(n-1)}{\chi_{1-\alpha/2, n-1}^2} s^2, \frac{(n-1)}{\chi_{\alpha/2, n-1}^2} s^2 \right) = \left(\frac{14}{31.32} 10.78, \frac{14}{4.07} 10.78 \right) = (4.82, 37.08)$$

and 99% CI for σ is

$$(\sqrt{4.82}, \sqrt{37.08}) = (2.20, 6.09).$$

2. Consider a sample of fat content of 100 randomly selected hotdogs. We have following observations: $\bar{x} = 22$, $S^2 = 10^2$. We are interested in checking if the fat content is 24.

```
> qnorm((1:10)/200)
[1] -2.57 -2.33 -2.17 -2.05 -1.96 -1.88 -1.81 -1.75 -1.6 -1.64
> qt((1:10)/200, 99)
[1] -2.63 -2.36 -2.20 -2.08 -1.98 -1.90 -1.83 -1.77 -1.71 -1.66
```

- (1) What is a test statistic and its distribution under the null assumption? [5pts].
- (2) What is the rejection region at level 0.01? Perform the test if the fat content is 24 at level 0.01 [5pts].
- (3) What is the type II error of the previous test when the actual fat content of hotdogs is 23? Express your answer using function Φ , which is defined as $\Phi(y) = \int_{-\infty}^y \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$ [5pts].

Solution. (1) The proper hypotheses are

$$H_0 : \mu = 24 \text{ vs. } H_1 : \mu \neq 24.$$

Since we have large sample ($n = 100$) with no assumption on the sample distribution, we use $Z = \bar{X} - 24$ as a test statistic. Under H_0 , $Z \sim N(0, 1)$. (2) We reject H_0 if $|\bar{X} - 24| > z_{\alpha/2} = z_{0.005} = -\text{qnorm}(0.005) = 2.57$. So the rejection region is $R = (-\infty, -2.57) \cup (2.57, \infty)$. Since $Z = 22 - 24 = -2 \notin R$, we do not reject H_0 . (3)

$$\begin{aligned} \beta(23) &= P(\text{not reject } H_0 | \mu = 23) = P(-2.57 < \bar{X} - 24 < 2.57 | \mu = 23) \\ &= P(-1.57 < \bar{X} - 23 < 3.57) \text{ when } \mathbb{E}X_i = 23 \\ &= \Phi(3.57) - \Phi(-1.57) \end{aligned}$$

3. When you toss a coin 100 times, you observed 40 heads.

- (1) Construct a 99% confidence interval for the probability of getting head [5pts].

(2) Test if the coin is biased at level 0.01 [5pts].

Solution. (1) Let p be the probability of getting head. Let X_i be the number of head when we toss the coin i -th time. Then $\hat{p} = \bar{x} = 0.4$.

$$\frac{\bar{X} - p}{\sqrt{\hat{p}(1 - \hat{p})/100}} \sim N(0, 1).$$

So the $100(1 - \alpha)\%$ CI for p is

$$\bar{x} \pm z_{\alpha/2} \sqrt{\hat{p}(1 - \hat{p})/100} = 0.4 \pm 2.57 \sqrt{0.4(1 - 0.4)/100} = 0.4 \pm 0.126$$