

Stat 312: Lecture 03

Minimum Variance Unbiased Estimator

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1. *Population* of interest is a collection of measurable objects we are studying. Let $X_1, \dots, X_n$ be a random sample from the population. Then sample mean $\bar{X}$ and sample variance S^2 are unbiased estimators of population mean μ and population variance σ^2 respectively.

Proof. Note that

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 = \frac{1}{n-1} \left[\sum_{i=1}^n X_i^2 - n\bar{X}^2 \right].$$

Then using the fact $\mathbb{E}(\bar{X})^2 = \mathbb{V}\bar{X} + (\mathbb{E}\bar{X})^2 = \sigma^2/n + \mu^2$, it can be shown that $\mathbb{E}S^2 = \sigma^2$.

2. There may be many unbiased estimators of θ . Given two unbiased estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ of θ . We choose one that gives less variance. If $\mathbb{V}(\hat{\theta}_1) \leq \mathbb{V}(\hat{\theta}_2)$, $\hat{\theta}_1$ is called more *efficient* than $\hat{\theta}_2$. An efficient estimator has less variability so we are more likely to make an estimate close to the true parameter value. The following coin flipping example clearly demonstrate this.

```
> a<-rbinom(1000,1,0.5)
> a
[1] 0 0 1 0 0 1 0 1 1 0 0 ...
> mean(a)
[1] 0.517
> mean(a[1:3])
[1] 0.3333333
> mean(a[1:9])
[1] 0.4444444
> mean(a[1:11])
[1] 0.3636364
```

3. Among all unbiased estimators, we choose the most efficient estimator called the *minimum variance unbiased estimator* (MVUE). The MVUE is an unbiased estimator with the smallest variance. MVUE is the most efficient estimator. An efficient estimator $\hat{\theta}$ will produce an estimate closer

to the true parameter θ . $\bar{X}$ is MVUE for μ (we will not prove this statement).

4. Given a random sample $X_1, \dots, X_n$, a linear estimator of parameter θ is an estimator of form

$$\hat{\theta} = \sum_{i=1}^n c_i X_i.$$

Then it can be shown that $\bar{X}$ is the MVUE for population mean μ among all possible linear estimators.

Proof. Case $n = 2$ will be proved. The general statement follows inductively. Consider linear estimators

$$\hat{\mu} = c_1 X_1 + c_2 X_2.$$

To be unbiased, $c_1 + c_2 = 1$. To be most efficient among all unbiased linear estimators, the variance has to be minimized. The variance is

$$\mathbb{V}\hat{\mu} = c_1^2 \mathbb{V}X_1 + c_2^2 \mathbb{V}X_2 = [c_1^2 + (1 - c_1)^2] \sigma^2$$

The quadratic term in the bracket $2c_1^2 - 2c_1 + 1$ is minimized when $c_1 = 1/2$.

Review Problems. You are not required to do these problems but these are problems you should be able to answer after each lecture. What is an unbiased estimator of population parameter μ^2 ? Exercise 6.3.