

Stat 312: Lecture 19

Least squares estimation

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Concepts

1. *Least squares method.* In the previous lecture, we studied the least squares estimation method for estimating parameters in linear regression. This method can be used to estimate other parameters in a different model. Given measurements $y_1, y_2, \dots, y_n$, they can be modeled as

$$Y_i = \mu + \epsilon_i,$$

where $\mathbb{E}\epsilon_i = 0$ and $\mathbb{V}\epsilon_i = \sigma^2$ with no assumption of normality. We are interested in estimating $\mu = \mathbb{E}Y$, the population mean. Let $\hat{\mu}$ be the estimator of μ . Then the predicted value is $\hat{y}_i = \hat{\mu}$ and the residual error is $r_i = y_i - \hat{y}_i$. So the total sum of the squared errors (SSE) is

$$SSE = \sum_{i=1}^n (y_i - \hat{\mu})^2.$$

To find minimum of SSE, we differentiate SSE with respect to $\hat{\mu}$ and get $\hat{\mu} = \bar{y}$.

2. *Weighted least squares method.* Suppose we have two population. Measurement are taken from the first population: $x_1, x_2, \dots, x_m$ and the second population: $y_1, y_2, \dots, y_n$. They are modeled as

$$X_i = \mu + \epsilon_i$$

with $\mathbb{E}\epsilon_i = 0$ and $\mathbb{V}\epsilon_i = \sigma_1$ while

$$Y_i = \mu + e_i$$

with $\mathbb{E}e_i = 0$ and $\mathbb{V}e_i = \sigma_2$. Errors ϵ_i and e_j are assumed to be independent. We can again use the least squares method to estimate the common population mean μ but this problem is more complicated than the previous case due to the different variabilities. We solve this problem by transforming the above equations to have the same variabilities:

$$\frac{X_i}{\sigma_1} = \frac{\mu}{\sigma_1} + \frac{\epsilon_1}{\sigma_1},$$

$$\frac{Y_i}{\sigma_2} = \frac{\mu}{\sigma_2} + \frac{e_1}{\sigma_2}.$$

Then $\frac{\epsilon_1}{\sigma_1}$ and $\frac{e_1}{\sigma_2}$ are i.i.d and SSE is given by

$$SSE = \frac{1}{\sigma_1^2} \sum_{i=1}^m (x_i - \hat{\mu})^2 + \frac{1}{\sigma_2^2} \sum_{i=1}^n (y_i - \hat{\mu})^2.$$

We again differentiate SSE with respect to $\hat{\mu}$. The non-trivial solution is

$$\hat{\mu} = \frac{1}{m/\sigma_1^2 + n/\sigma_2^2} \left[\frac{m}{\sigma_1^2} \bar{x} + \frac{n}{\sigma_2^2} \bar{y} \right].$$

This method is called the *weighted least squares* method.

Example Line probe measurements of mice micro-CT breast cancer imaging. Download data files from <http://www.stat.wisc.edu/~mchung/teaching/data/lineprobe/> into your computer directory.

```
> a<-read.table("d:/stat312/probe1.txt",header=T)
> a
  probe1
1 -403.30
2 -379.64
3 -359.26 ...
> probe1 <-a$probe1
> probe1
[1] -403.30 -379.64 -359.26 ...
> mean(probe1)
[1] -389.8323
> var(probe1)
[1] 1043.839
> mean(probe2)
[1] -388.046
> var(probe2)
[1] 812.3707
> mean(probe3)
[1] -385.6897
> var(probe3)
[1] 796.193
> mean(probe4)
[1] -437.0326
> var(probe4)
[1] 382.7925
```

For in this example, the least squares estimation for the population mean is -404.7 .

Homework 6. Due Dec 14 9:30AM. Solutions will be handed out exactly at 9:30AM. So no late homework will be accepted. 8.52, 9.2, 9.30, 9.36, 9.48, 12.16, 12.26.