

Stat 324: Lecture 03

Combinatorics

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1. Given n distinguishable objects $x_1, \dots, x_n$, *permutation* is a rearrangement of the n objects. There are $n!$ different rearrangements. *Combination* is a way to choose k distinct objects out of the n objects. There are $\binom{n}{k}$ different combinations. The notation is called n choose k .
2. *Binomial identity* $\sum_{k=0}^n \binom{n}{k} = 2^n$.
3. *Problem.* A fair coin is tossed independently n times ($n > 3$). Find the probability that we get 2 heads and $n - 2$ tails.

4. *Problem.* A fair coin is tossed independently n times ($n > 3$). Find the probability at least three of tosses yield heads.

Solution. Let X be the number of heads.

$$P(X \geq 3) = 1 - P(X < 3) = 1 - P(X = 0) - P(X = 1) - P(X = 2) \quad (1)$$

$$= 1 - \frac{1}{2^n} - \binom{n}{1} \frac{1}{2^n} - \binom{n}{2} \frac{1}{2^n}. \quad (2)$$

5. If n identical balls are placed at random into n distinct boxes, find the probability that exactly one box remains empty.

Solution. Let A be the empty box, B the box containing two balls and C 's $n - 2$ boxes containing one ball. There are $2\binom{n}{2}$ ways of arranging the letter sequence $ABC \dots C$. The size of the sample space is $\binom{n+n-1}{n}$ since it is equivalent to the number of ways of placing n 0's and $n - 1$ 1's in order. So the probability is $\frac{n(n-1)}{\binom{2n-1}{n}}$.

6. If n identical balls are placed into $n - 1$ distinct boxes so that each distinguishable arrangement is equally likely. Find the probability that no box remains empty.

Solution. Let A be the box containing two balls and B 's $n - 2$ boxes containing one ball. There are $n - 1$ ways of arranging the letter sequence $AB \dots B$. The size of the sample space is $\binom{n+n-2}{n}$ since it is equivalent to the number of ways of placing n 0's and $n - 2$ 1's in order. We expect the equal probability for each sample point. So the probability is $\frac{n-1}{\binom{2n-2}{n}}$.

Assigned Problems