

Stat 324: Lecture 04

Conditional Probability

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- For two events A and B with $P(B) > 0$, the *conditional probability* of A given that B has occurred is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

- The Smiths have two children. At least one of them is a boy. What is the probability that both children are boys?

Solution. Let X be the number of boys. Then

$$\begin{aligned} P(X = 2|X \geq 1) &= \frac{P(X = 2, X \geq 1)}{P(X \geq 1)} \\ &= \frac{P(X = 2)}{1 - P(X = 0)} = \frac{1/4}{1 - 1/4}. \end{aligned}$$

- The law of total probability.* For disjoint events A_1 and A_2 with $S = A_1 \cup A_2$,

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2).$$

- Bayes' theorem.*

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)}.$$

- 1/10 of men and 1/7 of women are color-blind. A person is chosen at random and that person is color-blind. What is the probability that the person is male. Assume males and females to be in equal numbers.

Solution. Let M=male, F=female, C=color-blind. Then

$$\begin{aligned} P(M|C) &= \frac{P(M \cap C)}{P(C)} \\ &= \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C|F)P(F)} \\ &= \frac{\frac{1}{10} \cdot \frac{1}{2}}{\frac{1}{10} \cdot \frac{1}{2} + \frac{1}{7} \cdot \frac{1}{2}}. \end{aligned}$$

- A box contains w white balls, b black balls or r red balls. A ball is chosen at random and if it is either black or red then it is replaced by a white ball and if it is white then it is replaced by a red ball. Now again draw a ball. What is the probability that the second ball drawn is red when the first ball drawn is red? What is the probability that the second ball drawn is white?

Solution. Let W_i, B_i, R_i be the event that i -th draw is a white, black and red ball respectively. The sample space is given by

$$S = W_1 \cup B_1 \cup C_1 = W_2 \cup B_2 \cup C_2.$$

$$P(R_2|R_1) = \frac{r-1}{w+b+r}.$$

$$\begin{aligned} P(W_2) &= P(W_2|W_1)P(W_1) \\ &+ P(W_2|B_1)P(B_1) + P(W_2|R_1)P(R_1) \\ &= \frac{w-1}{w+b+r} \frac{w}{w+b+r} \\ &+ \frac{w+1}{w+b+r} \frac{b}{w+b+r} \\ &+ \frac{w+1}{w+b+r} \frac{r}{w+b+r}. \end{aligned}$$

- Two events A and B are *independent* if $P(A|B) = P(A)$. Obviously A and B are independent if $P(A \cap B) = P(A)P(B)$.
- Suppose that $A \subset B$ and $P(A) > 0$ and $P(B) > 0$. Are two events A and B independent?
Solution. Since $A \subset B$, $P(A \cap B) = P(A)$. The condition for the independence is $P(A \cap B) = P(A)P(B)$. $P(B) = P(A)P(B)$ iff. $P(B) = 1$. Hence, if $P(B) = 1$, A and B are independent but if $P(B) < 1$, A and B are not independent.

Assigned Problems Exercise 2.60, Exercise 2.72
Read Chapter 3. **First MIDTERM EXAM will be on FEB 22. 9:30-10:45AM. There will be 4 problems in the midterm.**