

Stat 324: Lecture 06

Expectation

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1. The *expected value* of a discrete random variable X is defined as

$$\mathbb{E}X = \sum_x xP(X = x) = \sum_x xp(x).$$

Problem. The expectation of the geometric distribution is $1/p$.

2. The expected value of function $Y = h(X)$ is

$$\mathbb{E}h(X) = \sum_y yP(h(X) = y) = \sum_x h(x)P(h(X) = h(x)) = \sum_x h(x)P(X = x) = \sum_x h(x)p(x).$$

Based on this fact, it can be shown that $\mathbb{E}(aX + b) = a\mathbb{E}X + b$.

3. $\mathbb{V}X = \mathbb{E}X^2 - (\mathbb{E}X)^2$. $\mathbb{V}(aX + b) = a^2\mathbb{V}X$.
4. For $X \sim \text{Bernoulli}(p)$, $\mathbb{E}X = p$ and $\mathbb{V}X = p$.
5. Consider an experiment that give only two outcomes H or T . Let $P(H) = p$ and $P(T) = 1 - p$. This is a Bernoulli experiment. Now consider n independent experiments where each experiment gives only two outcomes. This is a *Binomial experiment*. Let X be the number of outcome H . This is called the *Binomial random variable* with parameter n and p . It will be denoted by $X \sim \text{Binomial}(n, p)$. Note that if X_i is the number of outcome H in the i -th experiment, $X = X_1 + X_2 + \cdots + X_n$.
6. *Problem.* Find a value c that minimizes $\mathbb{E}(X - c)^2$ for a discrete random variable X .
solution. Let $f(c) = \mathbb{E}(X - c)^2 = c^2 - (2\mathbb{E}X)c + \mathbb{E}X^2$. Then $f(c)$ is minimum at a point where $f'(c) = 0$. $f'(c) = 2c - 2\mathbb{E}X = 0$. So $c = \mathbb{E}X$ will give the minimum.
7. *Problem.* A man with n keys wants to open his door and tries the keys at random. Exactly one key will open the door. Find the mean and the variance of the number of trials if unsuccessful keys are eliminated.
solution. Let X be the number of trials. Then $P(X = 1) = \frac{1}{n}$, $P(X = 2) = \frac{n-1}{n} \frac{1}{n-1} = \frac{1}{n}$, $\dots$, $P(X = n) = \frac{n-1}{n} \dots \frac{1}{2} = \frac{1}{n}$. The random variable X is said to be *uniformly* distributed (or a discrete uniform random variable).

$$\mathbb{E}X = \sum_{x=1}^n \frac{x}{n} = \frac{1}{n} \sum_{x=1}^n x = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

$$\mathbb{E}X^2 = \sum_{x=1}^n \frac{x^2}{n} = \frac{1}{n} \frac{n(n+1)(2n+1)}{6}.$$

$$\text{So } \mathbf{Var}X = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}.$$

8. *Problem.* For $X \sim \text{Binomial}(n, p)$, compute $\mathbb{E}(X^3)$.

Solution. From p. 104 of your textbook, $\mathbb{E}X(X - 1) = n(n - 1)p^2$. Note that $\mathbb{E}X^3 = \mathbb{E}X(X - 1)(X - 2) + 3\mathbb{E}X^2 + 2\mathbb{E}X = n(n - 1)(n - 2)p^3$.

Assigned Problems Exercise 3.38., Exercise 3.50. Second Homework due on Feb 10. 9:30AM. Submit solutions to assigned problems from Lecture 4-7 (8 problems).