

Stat 324: Lecture 9

Normal distributions

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1. The $100(1 - \alpha)$ -th percentile of $N(0, 1)$ will be denoted by z_α . It is given by $100(1 - \alpha) = \Phi(z_\alpha) = \text{pnorm}(z_\alpha, 0, 1)$. We can find percentiles z_α using

$> z_\alpha = \text{qnorm}(1 - \alpha, 0, 1)$.

```
> alpha<-c(0,0.01,0.025,0.05,0.1)
> alpha
[1] 0.000 0.010 0.025 0.050 0.100
> qnorm(alpha,0,1)
[1] -Inf -2.33 -1.96 -1.64 -1.28
> qnorm(1-alpha,0,1)
[1]      Inf 2.33 1.96 1.64 1.28
```

2. The relationship between cdf and density function is $F(x) = \int_{-\infty}^x f(y) dy$. Given F , we can recover the density function by $f(x) = \frac{dF(x)}{dx}$ by the fundamental theorem of calculus.

Example. $f(x) = 2x$ for $0 < x < 1$ and $f(x) = 0$ otherwise. The cdf is given by $F(x) = x^2$ for $0 < x < 1$. $F(x) = 0$ for $x < 0$ and $F(x) = 1$ for $x > 1$. By differentiating F , we recover the density function.

3. Given $X \sim N(\mu, \sigma^2)$, let $Z = \frac{X - \mu}{\sigma}$. It can be shown that this is a standard normal distribution. First find the cdf of Z .

$$F_Z(z) = P\left(\frac{X - \mu}{\sigma} \leq z\right) = F_X(\sigma z + \mu).$$

By differentiating both sides,

$$f_Z(z) = f_X(\sigma z + \mu)\sigma = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

This is exactly the density function of $N(0, 1)$.

$$\mathbb{E}Z = (\mathbb{E}X - \mu)/\sigma = 0, \mathbb{V}Z = \mathbb{V}X/\sigma^2 = 1.$$

This implies that a linear function of a normal distribution is again a normal distribution.

4. *Problem.* The length of time required to complete a college test is found to be normally distributed with mean 50 minutes and standard deviation 12 minutes.

(a) When should the test be terminated if we wish to allow sufficient time for 90% of the students to complete the test? (b) What proportion of students will finish the test between 30 and 60 minutes? You can use the following R output.

Solution. Let X be the length of time to complete the test. Then $Z = \frac{X - 50}{12} \sim N(0, 1)$, the standard normal distribution. Need to find c such that $P(X < c) = P(Z < \frac{c - 50}{12}) = 0.9$. From the R output, $P(Z < 1.28) = \text{pnorm}(1.28, 0, 1) = 0.9$. $\frac{c - 50}{12} = 1.28$. So at least $c = 65.36$ minutes should be given. $P(30 < X < 60) = P(-1.67 < Z < 0.83) = \text{pnorm}(0.83, 0, 1) - \text{pnorm}(-1.67, 0, 1) = 0.75$.

5. $100(1 - \alpha)$ percentile of $N(\mu, \sigma^2)$ is $\sigma Z_\alpha + \mu$. This idea will be used in probability plots (Chapter 4.6).

Assigned Problems Exercises 4.46. Exercise 4.53.