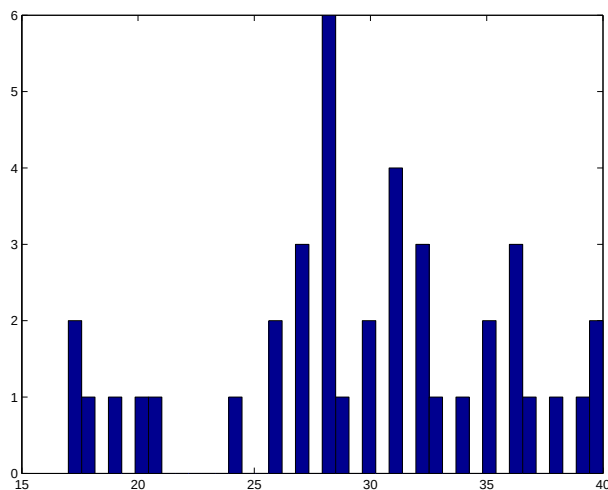


Grade distribution 29.65 ± 6.18 .



- (a) Suppose that A and B are independent events such that the probability that neither occurs is $\frac{1}{2}$ and the probability of B occurring is $\frac{1}{3}$. Determine the probability of A occurring.

Solution. $P((A \cup B)^c) = 1 - P(A \cup B) = 1/2$. So $P(A \cup B) = 1/2$. Since A and B are independent $P(A \cap B) = P(A)P(B) = P(A)/3$. By substituting terms into $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we have $1/2 = P(A) + 1/3 - P(A)/3$. Hence $P(A) = 1/4$.

- (b) If A and B are independent events, prove that A^c and B are also independent.

Solution. Since A and B are independent, $P(A \cap B) = P(A)P(B)$. We need show $P(A^c \cap B) = P(A^c)P(B)$. $P(B) = P(A \cap B) + P(A^c \cap B) = P(A)P(B) + P(A^c \cap B)$. Hence $P(A^c \cap B) = P(B) - P(A)P(B) = (1 - P(A))P(B) = P(A^c)P(B)$.

- A lie detector will show a positive reading (indicates a lie) 10% of the time when a person is telling the truth and 90% of the time when the person is lying. Suppose two people are suspects in a crime and only one is guilty. The guilty one will be lying while the innocent one will be telling the truth.

- What is the probability that the detector shows positive readings for both suspects?

Solution.

$$\begin{aligned}
 P(\text{positive}) &= P(\text{positive} \cap \text{lying}) + P(\text{positive} \cap \text{truth}) \\
 &= P(\text{positive}|\text{lying})P(\text{lying}) + P(\text{positive}|\text{truth})P(\text{truth}) \\
 &= 0.1 \times 0.5 + 0.9 \times 0.5 = 0.5
 \end{aligned}$$

(b) What is the probability that a person is lying when the detector shows a negative reading?

Solution.

$$\begin{aligned} P(\text{lying}|\text{negative}) &= \frac{P(\text{negative}|\text{lying})P(\text{lying})}{P(\text{negative})} \\ &= \frac{P(\text{negative}|\text{lying})P(\text{lying})}{P(\text{negative}|\text{lying})P(\text{lying}) + P(\text{negative}|\text{truth})P(\text{truth})}. \end{aligned}$$

$$P(\text{negative}|\text{lying}) = 1 - P(\text{positive}|\text{lying}) = 1 - 0.9 = 0.1.$$

$$P(\text{negative}|\text{truth}) = 1 - P(\text{positive}|\text{truth}) = 1 - 0.1 = 0.9. \text{ Hence}$$

$$P(\text{lying}|\text{negative}) = \frac{0.1 \cdot 0.5}{0.1 \cdot 0.5 + 0.9 \cdot 0.5} = 0.1.$$

3. A man with n keys wants to open his door and tries the keys at random. Exactly one key will open the door.

(a) Suppose unsuccessful keys are **not eliminated** from n keys. Determine the probability distribution of the number of trials until he opens the door.

Solution. A similar problem was solved during the review. Let X be the number of trials until he opens the door. Then $P(X = 1) = \frac{1}{n}$, $P(X = 2) = \frac{n-1}{n} \frac{1}{n}$, $P(X = 3) = (\frac{n-1}{n})^2 \frac{1}{n}$. In general, $P(X = i) = (\frac{n-1}{n})^{i-1} \frac{1}{n}$. So X is a geometric random distribution with $p = \frac{1}{n}$.

(b) Suppose unsuccessful keys are **eliminated** from n keys. Determine the probability distribution of the number of trials until he opens the door.

Solution. This is identical to the problem in one of the lecture notes. $P(X = 1) = \frac{1}{n}$, $P(X = 2) = \frac{n-1}{n} \frac{1}{n-1} = \frac{1}{n}$, $\dots$, $P(X = n) = \frac{n-1}{n} \dots \frac{1}{2} = \frac{1}{n}$. This type of distribution is called a discrete *uniform* distribution.

4. We have a probability density $f(x) = kx^2$ for $0 < x < 1$ and $f(x) = 0$ otherwise.

(a) Compute the variance and the expected value of the distribution.

Solution. $k = 3$ to make f a probability density. Note that $\mathbb{E}X = \int_0^1 3x^3 dx = \frac{3}{4}x^4|_0^1 = \frac{3}{4}$. $\mathbb{E}X^2 = \int_0^1 3x^4 dx = \frac{3}{5}x^5|_0^1 = \frac{3}{5}$. Hence $\mathbb{V}X = \frac{3}{5} - (\frac{3}{4})^2 = 0.0375$.

(b) Determine the 75-th percentile point.

Solution. Let η to be the 75-th percentile, then $\int_0^\eta 3x^2 dx = 0.75$. Then we have $\eta = 0.75^{1/3} = 0.9086$.

Assigned Problems Homework III due on Thursday March 3, 9:30AM. Do assigned problems in Lecture 07-10.