

2005 STAT 324 midterm 2 solution Moo K. Chung mchung@stat.wisc.edu

1. A box contains 7 white balls and 3 black balls. A ball is chosen at random and replaced by the opposite color. Now again draw a ball.

(a) What is the probability that the second ball drawn is white?

Solution. Let W_i be the event that the i -th ball is white and B_i be the event that the i -th ball is black.

$$\begin{aligned} P(W_2) &= P(W_2|W_1)P(W_1) + P(W_2|B_1)P(B_1) \\ &= \frac{6}{10} \frac{7}{10} + \frac{8}{10} \frac{3}{10} = 0.66 \end{aligned}$$

(b) What is the probability that the first ball drawn is black when the second ball drawn is white?

Solution. $P(B_1|W_2) = \frac{P(W_2|B_1)P(B_1)}{P(W_2)} = \frac{\frac{8}{10} \frac{3}{10}}{\frac{6}{10} \frac{7}{10} + \frac{8}{10} \frac{3}{10}} = \frac{12}{33}.$

2. The joint distribution of X and Y is given by

$$\begin{aligned} f(x, y) &= 2 \text{ for } y < x < 1, 0 < y < 1 \\ &0 \text{ otherwise.} \end{aligned}$$

(a) Compute the probability $P(\frac{1}{2} < X < 1, \frac{1}{2} < Y < 1)$.

Solution. Since it is a uniform distribution, the probability is equivalent to the area bounded by $\frac{1}{2} < X < 1$ and $\frac{1}{2} < Y < 1$ times 2 so it is $\frac{1}{4}$.

(b) Determine the marginal distributions of X and Y .

Solution. $f_X(x) = \int_x^1 2 \, dy = 2 - 2x$ for $0 < x < 1$. $f_Y(y) = \int_0^y 2 \, dx = 2y$ for $0 < y < 1$.

(c) Are X and Y independent? Prove your statement.

Solution. $f(\frac{3}{4}, \frac{3}{4}) \neq f_X(\frac{3}{4})f_Y(\frac{3}{4})$. Hence they are not independent.

3. A random sample $X_1, X_2, \dots, X_n$ comes from the distribution

$$f(x) = \lambda x^{\lambda-1}, 0 < x < 1.$$

(a) Write down the likelihood function for λ clearly.

Solution. $L(\lambda) = \lambda^n (X_1 \cdot X_2 \cdots X_n)^{\lambda-1}.$

(b) Derive the maximum likelihood estimator of θ .

Solution. The log-likelihood is $\ln L(\lambda) = n \ln \lambda + (\lambda - 1) \sum_{i=1}^n \ln X_i$. Differentiating with respect to λ , we get $\frac{\partial \ln L(\lambda)}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \ln X_i = 0$. Hence $\hat{\lambda} = -\frac{n}{\sum_{i=1}^n \ln X_i}.$

4. The intelligence quotient (IQ) of 10 dogs are measured:

25, 21, 22, 17, 29, 25, 16, 20, 19, 22.

The measurements are assumed to follow a normal distribution with known variance 9.

The following R output is available.

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> qnorm(c(0.025, 0.05, 0.1))  
[1] -1.96 -1.64 -1.28
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(a) Construct a 95% confidence interval of the mean IQ.

Solution. Since $\sigma = 3$ is known, a CI is based on the Z -statistic: $z = (\bar{x} - \mu)/(\sigma/\sqrt{n})$. CI is given by $\bar{x} \pm z_{\alpha/2}3/\sqrt{10}$. The final answer is $19.7 \leq \mu \leq 23.5$.

(b) Determine the minimum sample size needed for the width of a 95% confidence interval to be less than 2.

Solution. We need $2z_{\alpha/2}\frac{3}{\sqrt{n}} \leq 2$. Solving this, we need $(1.96 \cdot 3)^2 = 34.57 \leq n$. Hence $n = 35$.