

STAT 324 Sample midterm problems

This was actual exam problems in year 2001.

1. If A and B are independent events, with $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, find the following:
(a) $P(A^c \cap B^c)$

Solution 1. Since A and B are independent, A^c and B^c are independent. So $P(A^c \cap B^c) = P(A^c)P(B^c) = (1 - P(A))(1 - P(B)) = (1 - \frac{1}{3})(1 - \frac{1}{4}) = \frac{1}{2}$.

Solution 2. $P(A \cap B) = P(A)P(B) = \frac{1}{12}$. $P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B)) = 1 - (\frac{1}{3} + \frac{1}{4} - \frac{1}{12}) = \frac{1}{2}$.

- (b) $P(A^c|B)$.

Solution. $P(A^c|B) = 1 - P(A|B) = 1 - \frac{P(A \cap B)}{P(B)} = \frac{2}{3}$.

2. The length of time required to complete a college test is found to be normally distributed with mean 50 minutes and standard deviation 12 minutes.
(a) When should the test be terminated if we wish to allow sufficient time for 90% of the students to complete the test? (b) What proportion of students will finish the test between 30 and 60 minutes? You can use the following R output.

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> pnorm(c(1.25, 1.26, 1.27, 1.28, 1.29))  
[1] 0.8943502 0.8961653 0.8979577 0.8997274 0.9014747
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Solution. Let X be the length of time to complete the test. Then $Z = \frac{X-50}{12} \sim N(0, 1)$, the standard normal distribution. Need to find c such that $P(X > c) = P(Z > \frac{c-50}{12}) = 0.1$. From the R output, $P(Z > 1.28) = 0.1$. From the symmetry, $\frac{c-50}{12} = 1.28$. So at least $c = 65.36$ minutes should be given. $P(30 < X < 60) = P(-1.67 < Z < 0.83) = 1 - P(Z > 1.67) - P(Z > 0.83) = 1 - 0.0475 - 0.2033 = 0.75$.

3. In a bolt factory, machines 1, 2 and 3 respectively produce 20 %, 30% and 50% of the total output. Of their output, 5%, 3% and 2% are defective. A bolt is selected at random.
(a) List the sample space for this experiment and assign a probability to each sample point.

Solution. Let M_1, M_2, M_3 be the events that the selected bolt comes from machines 1, 2 and 3 respectively. Then the sample space is given by $S = M_1 \cup M_2 \cup M_3$ with $P(M_1) = 0.2, P(M_2) = 0.3, P(M_3) = 0.5$. A slightly different partitioning of the

- (b) What is the probability that it is defective?

Solution. Let D be the event that the bolt is defective. From the law of total probability,

$$P(D) = P(D|M_1)P(M_1) + P(D|M_2)P(M_2) + P(D|M_3)P(M_3),$$

where $P(D|M_1) = 0.05$, $P(D|M_2) = 0.03$, $P(D|M_3) = 0.02$. Hence, $P(D) = 0.029$.

(c) Given that it is defective, what is the probability that it was made by machine 1?

Solution. $P(M_1|D) = \frac{P(D \cap M_1)}{P(D)} = \frac{P(D|M_1)P(M_1)}{P(D)} = 0.52$

4. A box contains 100 white and 200 black balls. Balls are randomly selected, one at a time, until a black one is obtained. Assume that each selected ball is put back into the box before the next one is drawn.

(a) What is the probability that exactly n draws are needed?

Solution. Let X be the number of draws. Then $P(X = n) = (\frac{1}{3})^{n-1} \frac{2}{3}$. This is a geometric distribution with parameter $p = \frac{2}{3}$.

(b) What is the probability that at least 10 draws are needed?

Solution.

$$\begin{aligned} P(X \geq 10) &= \sum_{n=10}^{\infty} P(X = n) \\ &= \sum_{n=10}^{\infty} \left(\frac{1}{3}\right)^{n-1} \frac{2}{3} = \frac{2}{3} \left(\frac{1}{3}\right)^9 \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \\ &= \frac{2}{3} \left(\frac{1}{3}\right)^9 \frac{1}{1 - 1/3} = \left(\frac{1}{3}\right)^9. \end{aligned}$$

(c) What is the expected number of draws until a black ball is obtained?

Solution. $\mathbb{E}X = \sum_{x=1}^{\infty} x \left(\frac{1}{3}\right)^{x-1} \frac{2}{3} = \sum_{x=1}^{\infty} x (1-p)^{x-1} p = \frac{1}{p} = 3/2$.

5. Let X have the probability density function $f(x) = ce^{-x}$ for $1 < x < \infty$ and $f(x) = 0$ otherwise.

(a) Find the value c .

Solution. $\int_1^{\infty} ce^{-x} dx = -ce^{-x}|_1^{\infty} = ce^{-1} = 1$. So $c = e$.

(b) Find the cumulative distribution function of X and plot its graph.

Solution. Integrating f , we get $F(x) = 0$ for $x \leq 1$ and $F(x) = 1 - e^{1-x}$ for $x > 1$. F is a nondecreasing function and $F(\infty) = 1$.

(c) Find $P(-1 < X \leq 3 | X \geq 3)$.

Solution. $P(-1 < X \leq 3 | X \geq 3) = \frac{P(X=3)}{P(X \geq 3)} = 0$ since the probability that a continuous random variable taking a single point is zero.