

Stat 471: Lecture 7

Generalized Inverse.

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1.
$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
3. Computing the generalized inverse is based on the singular value decomposition (SVD). Let X be $n \times p$ matrix with $n \geq p$. Then SVD of X is

$$X = UDV',$$

There is no exact solution but it is possible to get a solution in the least-squares sense. Note that

$$2y_1 + y_2 - 1 = \epsilon_1, 4y_1 + y_2 - 1 = \epsilon_2.$$

Minimize the sum of squared errors, $\min(X\beta - c)'(X\beta - c)$, where $c = (1, 1)'$, $\beta = (y_1, y_2)'$ and $X = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$. $\frac{\partial}{\partial \beta} \beta' A = A$, $\frac{\partial}{\partial \beta} \beta' A \beta = 2A\beta$.

Then it can be shown that $\hat{\beta} = (X'X)^{-1}X'c$. The generalized inverse (psudo inverse) of X is

$$X^- = (X'X)^{-1}X'.$$

2. Moore-Penrose generalized inverse X^- is a unique matrix that satisfies the following matrix equations

$$XX^-X = X, X^-XX^- = X^-$$

$$(XX^-)' = XX^-, (X^-X)' = X^-X.$$

(see C.R. Rao and S.K. Mitra, Generalized inverse of matrices and its applications, Wiley, New York, 1971; Penrose, R., A Generalized Inverse for Matrices. Proc. Cambridge Phil. Soc. 51, 406-413, 1955.)

```
>>X=[2 1
      4 2]
>>Y=pinv(X)
Y =
    0.0800    0.1600
    0.0400    0.0800
>> X*Y*X
ans =
    2.0000    1.0000
    4.0000    2.0000
```

```
>> [U,D,V]=svd(X)
D =
    5.0000         0
         0    0.0000
>> U*D*V'
ans =
    2.0000    1.0000
    4.0000    2.0000
>> V*[1/5 0
      0 0]*U'
ans =
    0.0800    0.1600
    0.0400    0.0800
```

where $U_{n \times p}$ has orthonormal columns, $D_{p \times p} = \text{Diag}(d_1, \dots, d_p)$ is diagonal with non-negative elements and $V_{p \times p}$ is orthogonal. For symmetric positive definite matrix $X_{p \times p}$, we have diagonalization $X = QDQ'$ (lecture 07). For nonsingular matrix $X_{n \times n}$, $X = UDV'$ and U is orthogonal. Let $D^- = \text{Diag}(d_1^-, \dots, d_p^-)$, $d_i^- = 1/d_i$ if $d_i \neq 0$ and $d_i^- = 0$ if $d_i = 0$. Then it can be shown that the Moore-Penrose generalized inverse is given by

$$X^- = VD^-U'.$$