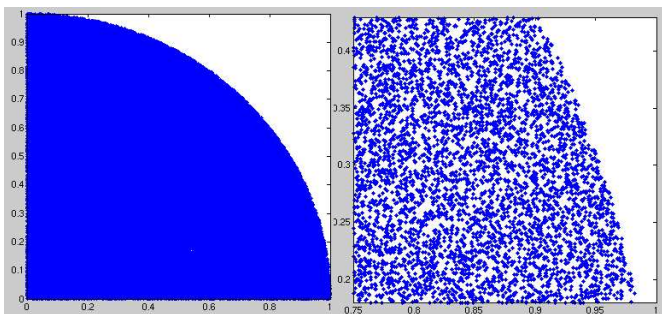


Stat 471: Lecture 10

Monte-Carlo Integral.

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September 29, 2003



1. The constant π is usually computed by summing up infinite series, i.e.

$$\pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right).$$

There is a slightly different way to estimate the digits of π based on a computer simulation. Let A be the area of a quarter circle with unit radius bounded by a unit square B . Suppose we throw darts at the square at random. Define binary random variable $X = 1$ if a dart hits A and $X = 0$ if it hits A^c . Note that

$$\mathbb{E}X = \int_A dx dy = \frac{\pi}{4}.$$

Note that the sample mean $\bar{X}_n = \sum_{i=1}^n X_i/n$ converges to $\mathbb{E}X$ as $n \rightarrow \infty$. So

$$\pi = 4 \lim_{n \rightarrow \infty} \bar{X}_n.$$

```
x=rand(100000,1);y=rand(100000,1);
rsquare=x.^2+y.^2;
accept=find(rsquare <1);
>> accept'
ans =
    2    3    4    5    6    7    8   10   11   15   ...
plot(x(accept),y(accept),'.')
>>4*size(accept,1)/1000000
ans =
    3.1336
```

2. There are many Monte-Carlo techniques for evaluating integrals. The above method is based on generating specific random numbers. In general if we know how to generate random variable $X \sim f$, we can generate $g(X)$. So

$$\mathbb{E}g(X) = \int g(x)f(x) dx.$$

$\mathbb{E}g(X)$ can be approximated by the sample mean $\bar{g}(X) = \sum_{i=1}^n g(X_i)/n$. What if you want to compute $\int g(x) dx$? Note that

$$\int g(x) dx = \int \frac{g(x)}{f(x)} f(x) dx = \mathbb{E} \frac{g(X)}{f(X)}.$$

So we generate random numbers $g(X)/f(X)$.

3. Empirical distribution. Estimate the cumulative distribution of the standard normal using the Monte-Carlo integration method.

$$\Phi(x) = \int_{-\infty}^x \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx.$$

Let g be a index function defined as

$g(y) = \mathbb{I}_x(y) = 0$ if $y \leq x$ and 1 if $y > x$. Then $\Phi(x) = \mathbb{E} \mathbb{I}_x(Z)$, where $Z \sim N(0,1)$. For given x , $\mathbb{E} \mathbb{I}_x(Z)$ is estimated by the proportion of the random numbers that is less than x . This is true for an arbitrary cumulative distribution function and we call this Monte-Carlo estimate as the empirical (cumulative) distribution.

Interesting paper by John Von Neumann (1951): Various techniques used in connection with random digits. Proceedings of Symposium on Monte Carlo Method (held June. July, 1949, in Los Angeles), chapter 13. Summary written by G. E. Forsythe, Journal of Research of the National Bureau of Standards **12** 36-38. Next lecture topic is about variance reduction in MC.