

Stat 471: Lecture 11

Importance Sampling.

Moo K. Chung
mchung@stat.wisc.edu

October 1, 2003

1. Estimate

$$p = \int_{-\infty}^{-2} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx.$$

We use the formula

$$\mathbb{E}g(X) = \int g(x)f(x) dx$$

with $g(x)$ to be an index function such that $g(x) = 1$ if $x \in (-\infty, -2)$ and 0 otherwise. Let $X \sim N(0, 1)$. Then $\hat{p} = \sum_{i=1}^n g(X_i)/n$.

```
n=1000000
X=snorm(n);
index=find(X<-2);
g=zeros(n,1);
g(index)=1;
>>[mean(g) var(g)]
ans =
    0.0228    0.0223
>>normcdf(-2)
ans =
    0.0228
```

2. An alternate approach to estimate p would be to use the method of *importance sampling*

$$\mathbb{E}g(X) = \int g(x) \frac{f(x)}{h(x)} h(x) dx = \mathbb{E} \left[g(X) \frac{f(X)}{h(X)} \right]$$

for some distribution h . A more proper name for this technique would be *weighted sampling*. Then we estimate the integral with $\hat{p} = \sum_{i=1}^n w(X_i)g(X_i)/n$ with the importance sampling weights $w(X_i) = f(X_i)/h(X_i)$.

3. The aim of importance sampling is to sample more frequently from values by weighting more so that the contribution to the integral is the greatest (We will probably talk about *importance resampling* in the context of bootstrap). How do we choose h ?

We choose h such that $\mathbf{Var}(\hat{p})$ is minimal. For the importance sampling estimator $\hat{p}$,

$$\mathbf{Var}(\hat{p}) = \mathbf{Var}[g(X_i)f(X_i)/h(X_i)]/n$$

Now note that if $h(x) = cg(x)f(x)$ for some constant c , the variance vanishes. Hence this choice of h must be the minimizer. Now the only requirement is that h need to be a density. It is satisfied if

$$h(x) = \frac{|g(x)|f(x)}{\int |g(y)|f(y) dy}.$$

See C.P. Robert and G. Casella, Monte Carlo Statistical Method for Jensen's inequality based proof. But this is somewhat useless! We do not know what the value of $\int |g(y)|f(y) dy$ is! You can't apply importance sampling blindly. In next lecture we will study how to choose h .

4. See Rubinstein, R.Y. (1981) Simulation and the Monte Carlo Method for details on the importance sampling.