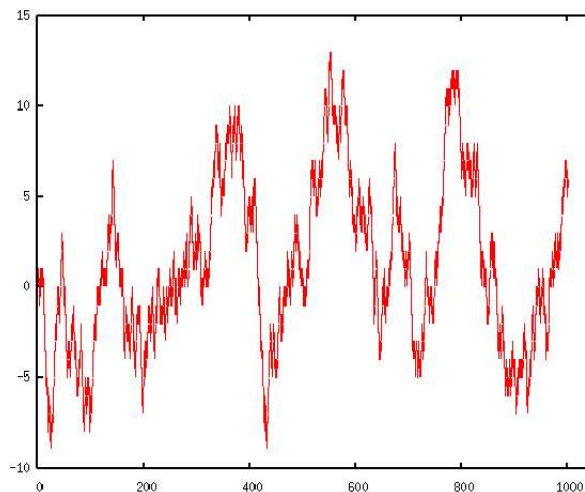


Stat 471: Lecture 18

Markov Chains

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kernel is given by

$$P(X_{i+1} = x + 1 | X_i = x) = \frac{1}{2},$$

$$P(X_{i+1} = x - 1 | X_i = x) = \frac{1}{2}.$$

We can model the coin tossing a simple random walk $X_{i+1} = X_i + \epsilon_i$, with $P(\epsilon_i = 1) = \frac{1}{2}$ and $P(\epsilon_i = -1) = \frac{1}{2}$. The expected pay off does not change over time, i.e. $\mathbb{E}X_{i+1} = \mathbb{E}X_i$.

```
x(1)=0; n=1000
e = (-1).^(rand(n,1) < 0.5)
for i=1:(n+1)
    x(i+1)=x(i) + e(i);
end;
```

1. A stochastic process is a collection of random variables X_i indexed over some set A , i.e. $\{X_i : i \in A\}$. If A is a discrete set, we have a discrete stochastic process. If A is continuous, we have a continuous stochastic process. Let $X_0, X_1, \dots$ be a stochastic process. A Markov chain is a sequence of random variables such that the next state X_{i+1} depends only on the current state X_i (memoryless property), i.e.

$$P(X_{i+1} = y | X_i = x_i, \dots, X_0 = x_0) \\ = P(X_{i+1} = y | X_i = x_i).$$

The probability $P_{xy} = P(X_{i+1} = y | X_i = x)$ is called the *transition kernel*. Note that $\sum_y P_{xy} = 1$. The initial distribution for X_0 determine the distribution for any n -th state.

In the case of random walk, $X_{i+1} = \sum_{j=0}^n \epsilon_j + X_0$. Then $P(X_{n+1} = x | X_0 = 0) = P(\sum_{j=0}^n \epsilon_j = x)$. Compute this probability (HW 4).

Chapman-Kolmogorov equation. Let P_{xy}^n be the transition kernel from x to y in n -steps, i.e. $P_{xy}^n = P(X_{i+n} = y | X_i = x)$. Then we have

$$P_{xy}^{m+n} = \sum_z P_{xz}^m P_{zy}^n.$$

The the probability of reaching y from x in n -steps is the sum of all probabilities going from x to y through an intermediate point z . Let $\mathbf{P}^n = (P_{ij}^n)$ be a matrix. Then in terms of matrix, Chapman-Kolomogorov equation can be trivially written as $\mathbf{P}^{m+n} = \mathbf{P}^m \mathbf{P}^n$.

Random walk. You play a coin tossing game with me. You win \$1 if a head appears and lose \$1 if a tail appears. Let X_i be your total gain after i tosses. Assume the coin to be unbiased. Then the transition