

Stat 471: Lecture 19

Markov Chains II.

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October 22, 2003

1. A rat became insane and moves back and forth between position 1 and 2. Let X_i be the position of the rat at the i -th move. Suppose that the transition probability is given by

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}.$$

A state i is called *recurrent* if the Markov chain returns to i with probability 1 in a finite number of steps otherwise the state is *transient*. The recurrent of a state is equivalent to a guarantee of a sure return. Is the state 1 and 2 recurrent?

$$\sum_{i=2}^{\infty} P(X_i = j | X_0 = j, X_1 \neq j, \dots, X_{i-1} \neq j)$$

$$= \frac{1}{2} + \frac{1}{2^2} + \dots = 1$$

for $j = 1, 2$. So the state 1 and 2 are recurrent.

2. Let us find $P(X_n = 1 | X_0 = 2)$. Note that this is n step transition probability denoted by P_{12}^n . It is computed by the Chapman-Kolmogorov equation. For this we need to compute $\mathbf{P}^n$. In general you can compute this by the Cayley-Hamilton theorem and solving n -th order finite difference equation. But why don't we skip it but basically you can compute it numerically in MATLAB.

```
P=[1/2 1/2
    1 0]
>>P^2
ans =
    0.75000    0.25000
    0.50000    0.50000
>> P^5
ans =
    0.65625    0.34375
    0.68750    0.31250
>> P^100
```

```
ans =
    0.66667    0.33333
    0.66667    0.33333
>> P^1000000000
ans =
    0.66667    0.33333
    0.66667    0.33333
```

So we can see that the transition probabilities converge: $\lim_{n \rightarrow \infty} P_{ij}^n = \pi_j$. $\pi_1 = \frac{2}{3}$ and $\pi_2 = \frac{1}{3}$.

3. A Markov chain is stationary or homogeneous if $(X_0, \dots, X_i) \sim (X_n, \dots, X_{n+i})$. Suppose we have state space S as the sample space and let $\pi = \{\pi_j : j \in S\}$ be a probability distribution defined on S , i.e. $\sum_{j \in S} \pi_j = 1$. Then π is a stationary distribution for the Markov chain with transition probability $\mathbf{P}$ if $\pi' = \pi \mathbf{P}$. Note that

$$\pi' = \pi' \mathbf{P} = \pi' \mathbf{P}^2 = \dots = \pi' \mathbf{P}^n.$$

This probability π is also called *invariant probability measure*. Read S.I. Resnick's Adventures in Stochastic Processes for theoretical details on Markov Chains and a stationary distribution. A couple of interesting properties on π_j . $\frac{1}{\pi_j}$ is the expected number of states a Markov chain should take to return to the original state j . In our crazy rat example, the rat will return to position 2 in average 3 steps if it was at the position 2 initially.

4. An increased stability for a Markov chain can be achieved if we let $X_0 \sim \pi$, i.e. $P(X_0 = j) = \pi_j$. From the total law of probability $P(A) = \sum_{i=1}^{\infty} P(B_i)P(A|B_i)$ if $\{B_i\}$ partition S . So $P(X_1 = j) = \sum_{i \in S} \pi_i P_{ij}$. Hence $X_1 \sim \pi' \mathbf{P} = \pi'$. Similarly $X_i \sim \pi$. MCMC is based on this fact and requirement.