Stat 471: Lecture 22
MCMC II.

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1. *Random-walk Metropolis.* The Metropolis sampler refers to the original version of Metropolis *et al.* (1953). For proposal distribution, we assume $q(y|x) = q(|y-x|)$. See example $Y \sim N(x,1)$ in lecture 20. In this case,

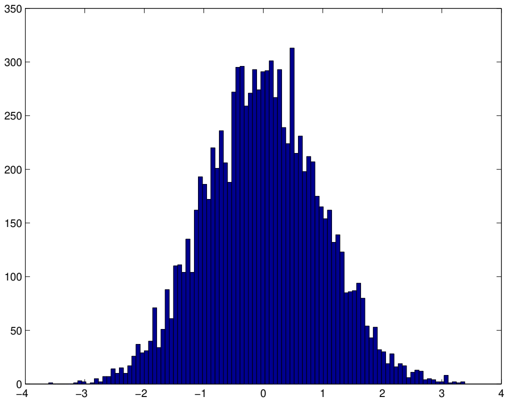
   $$\alpha(X_i, Y) = \min\Big(\frac{\pi(Y)}{\pi(X_i)}, 1\Big).$$

   Given $X_i = x_i$,
   1. Generate $Y \sim g(|y - x_i|)$.
   2. $X_{i+1} = Y$ with probability $\alpha(X_i, Y)$ and $X_{i+1} = X_i$ otherwise.

   Generating $Y \sim g(y - x_i)$ is equivalent to $Y = X_i + \epsilon_i$, where $\epsilon_i \sim g(y)$. So we have random walk $X_{i+1} = X_i + B_i\epsilon_i$ where B_i is Bernoulli with $P(B = 1) = \alpha$.

   Let's generate $N(0,1)$ from $Unif(-2,2)$.

```
n=10000;
x(1)=0;
snrpdf=inline(’exp(-x^2/2)’);
for i=2:n
  y=x(i-1) + 2*(1-2*rand);
  alpha=min([1,snrpdf(y)
       /snrpdf(x(i-1))]);
  u=rand;
  if u <= alpha
    x(i)=y;
  else
    x(i)=x(i-1);
  end
end
```



2. *Independent sampler* is the special case of H-M algorithm (Tierney, 1994). Proposal distribution q is independent of x, i.e. $q(y|x) = q(y)$. In this case

   $$\alpha(X_i, Y) = \min\Big(\frac{\pi(Y)}{\pi(X_i)}\frac{q(X_i)}{q(Y)}, 1\Big).$$

   This method does may not work well if q is very similar to π. Given X_i,
   1. Generate $Y \sim q(y)$.
   2. $X_{i+1} = Y$ with probability $\alpha(X_i, Y)$ and $X_{i+1} = X_i$ otherwise.

   The independent sampler can be compared to Accept-reject method for generating random numbers. In AR method,
   1. Generate $Y \sim q(y)$
   2. $X = Y$ with probability $\frac{\pi(Y)}{cq(Y)}$.

   It can be shown that the independent Metropolis-Hastings algorithm is more efficient than the Accept-Reject method since independent M-H will accept more proposed values(see Robert and Casella, Monte Carlo Statistical Method for detail).

3. The *Gibbs sampler* is a special case of the Metropolis-Hastings algorithm but it differ in two ways:
   1. We always accept Y (candidate point).
   2. Need to know full conditional distributions.

   By construction, the Gibbs sampler is multidimensional. Here we will only consider 2-dimensional case. Given joint density $f(x,y)$, the Gibbs sampler is a way to generate the marginal density $f_X(x) = \int f(x,y)\,dy$ and $f_Y(y) = \int f(x,y)\,dx$.

   We will study the Gibbs sampler in lecture 23.