Stat 471: Lecture 23
Gibbs Sampler.

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1. The Gibbs sampler was developed by Geman & Geman (1984). Let $\mathbf{Z_i} = (X_i, Y_i)'$ be a Markov chain. The Gibbs sampler can be used to generate specific multivariate distributions. Let $f(x, y)$ be a given joint density and $f(x|y)$ and $f(y|x)$ to be conditional densities. The Gibbs sampling algorithm is given by

   1. Generate $\mathbf{Z}_0 = (X_0, Y_0)'$. Set $i = 1$.
   2. Generate $X_i \sim f(x_i|Y_{i-1} = y_{i-1})$
   3. Generate $Y_i \sim f(y_i|X_i = x_i)$
   4. Set $i = i + 1$ and goto step 2.

2. **Theorem.** $X_i, Y_i, (X_i, Y_i)$ generated by the Gibbs sampler are Markov chains.
   Proof. For chain X_i, the transition kernel is

   $$K_X(x, x^*) = \int f(y|x)f(x^*|y)\,dy.$$

   Since X_i is simulated conditionally only on x_{i-1}, $\{X_i\}$ is a Markov chain. The invariant distribution of X_i is marginal distribution $f_X(x) = \int f(x, y)\,dy$.

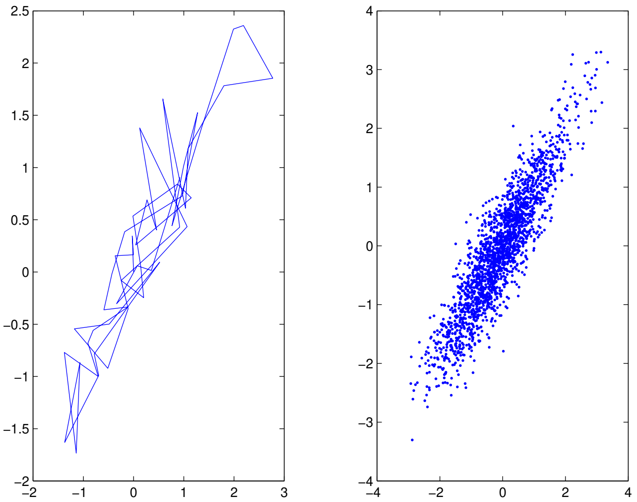
3. Let us simulate bivariate normal

   $$\mathbf{Z} = (X, Y)' \sim N(0, \left[\begin{array}{cc} 1 & \rho \\ \rho & 1 \end{array}\right])$$

   The joint density for $\mathbf{Z} \sim N(\mu, V)$ is given by

   $$f(\mathbf{z}) = \frac{1}{2\pi \det V^{1/2}} \exp\Big[-\frac{\mathbf{z}V^{-1}\mathbf{z}}{2}\Big].$$

   To simulate bivariate normal using the Gibbs sampler, we need to know the conditional distribution (we already know how to simulate multivariate normal based on the Choleskly factorization).



Then it can be shown that

$$f(x|y) = \frac{f(x, y)}{f(y)} = \frac{f(\mathbf{z})}{\int f(\mathbf{z})\,dx}.$$

It can be shown that

$$X|Y = y \sim N(\rho y, 1-\rho^2) \sim \rho y + \sqrt{1-\rho^2}N(0, 1),$$

$$Y|X = x \sim N(\rho x, 1-\rho^2) \sim \rho x + \sqrt{1-\rho^2}N(0, 1).$$

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x(1)=0;y(1)=0;
rho=0.9;
c=sqrt(1-rho^2);
for i=2:6000
   x(i)=rho*y(i-1)+ c*snrnd(1);
   y(i)=rho*x(i)+ c*snrnd(1);
end;
subplot(1,2,1);
plot(x(1:50),y(1:50));
subplot(1,2,2);
plot(x(4000:6000),y(4000:6000),'.');
```