

Stat 471: Lecture 33

Gaussian Processes

Moo K. Chung mchung@stat.wisc.edu

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1. In lecture 28 we simulated signal of the form

$$Y_i = g(p_i) + \epsilon_i \quad (1)$$

for the bivariate smoothing estimation of g . We assumed ϵ_i to be i.i.d. $N(0, \sigma^2)$.

```
[px,py] = meshgrid([-3:0.1:3]);
g = px.*exp(-px.^2-py.^2);
e=normrnd(0,1,61,61);
y1=g+0.2*e;
```

The above error assumption is unrealistic since most observations y_i are correlated. The reasonable assumption would be that the covariance between Y_i and Y_j would be a function of distance between p_i and p_j , i.e.

$$\text{Cov}(Y_i, Y_j) \propto \rho(\|p_i - p_j\|)$$

for some function ρ . One way to simulate correlated noise ϵ_i is based on Gaussian random processes.

2. A random process $\epsilon(t)$ is a Gaussian process if $\epsilon(t_1), \dots, \epsilon(t_m)$ are multivariate normal for any $t_i \in \mathbb{R}^N$. A mean zero Gaussian random variable is completely characterized by variance. Similarly mean zero Gaussian process is characterized by the *covariance function* R which is defined as $R(t, s) = \mathbb{E}[\epsilon(t)\epsilon(s)]$. Sometimes it is called the *autocovariance function*. Note that $R(t, t)$ gives the variance of $\epsilon(t)$. If we specify $R(t, s)$, we know the corresponding Gaussian process ϵ . Based on Gaussian process, we rewrite equation (1),

$$Y(t) = g(t) + \epsilon(t).$$

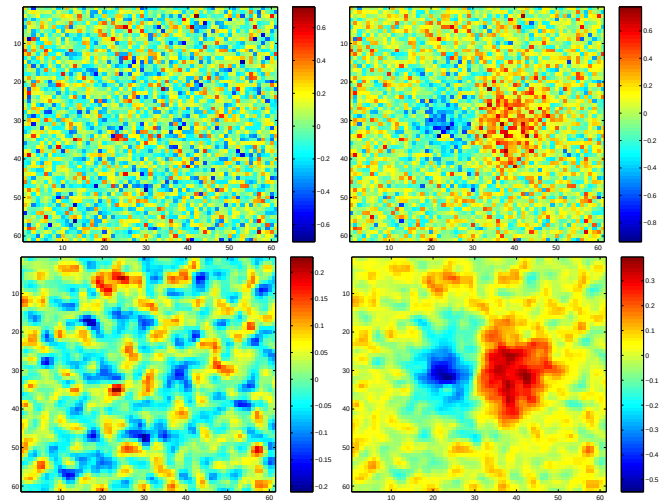
3. *Linear filtering* of a random process w is defined as

$$\epsilon(t) = \int K(t, \tau)w(\tau)d\tau$$

where K is the kernel of the linear filter. In many applications, we are interested in the kernel of form $K(t, \tau) = K(t - \tau)$ and in such case the above linear filtering is called the *convolution*. When $K(t - \tau) \sim N(0, \sigma^2 I_N)$, we have Gaussian kernel smoothing of process w , i.e.

$$\epsilon(t) = K * w(t).$$

Sometime this is called the *moving average* (MA) technique for obvious reason.



4. *White noise* is defined as a random process whose covariance function is proportional to the Dirac-delta function δ , i.e. $R(t, s) \propto \delta(t - s)$. We can define it as the limiting density function of $N(0, \sigma^2 I_N)$ as $\sigma \rightarrow 0$. Sometimes this is taken as the definition of Dirac-delta. *White Gaussian noise* is a white noise whose linear filtering is a Gaussian process. This is the usual definition of white Gaussian noise that can be used in simulating Gaussian processes. In practice, we use the discrete white Gaussian noise which is simply $N(0, \sigma^2 I_N)$ random variables. The discrete version of kernel smoothing is

$$\epsilon(t) = \sum_{i=1}^m K(t - \tau_i)w(\tau_i)$$

where τ_i are regular grid points in $\mathbb{R}^N$. Assume further that $w(\tau_i) \sim N(0, \sigma^2 I_N)$ and $w(\tau) = 0$ if $\tau \neq \tau_i$. Since any linear combination of Gaussian is again Gaussian, $\epsilon(t_1), \dots, \epsilon(t_m)$ must be multivariate normal. Therefore, process $\epsilon(t)$ constructed in this way must be a Gaussian process.

```
kernel=inline('exp(-(x.^2+y.^2)/(2*sigma^2))')
dx=kron(ones(5,1),[-2:2]);
dy=kron(ones(1,5),[-2:2]');
weight=kernel(0.8,dx,dy)/
    sum(sum(kernel(0.8,dx,dy)))
smooth_e=conv2(e, weight,'same');
y2=g+0.2*smooth_e;
```