Stat 992: Lecture 01
Gaussian Random Fields.

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1. *Spatiotemporal model.* Suppose we can measure temperature Y at position x and time t in a classroom $M \in \mathbb{R}^3$. Since every measurement will be error-prone, we model the temperature as

$$Y(x,t) = \mu(x,t) + \epsilon(x,t)$$

where μ is the real unknown signal and ϵ is measurement error. The measurement error can be modelled as a random variable. So at each point $(x,t) \in M \otimes \mathbb{R}^+$, measurement error $\epsilon(x,t)$ is a random variable. The collection of random variables

$$\{\epsilon(x,t) : (x,t) \in M \otimes \mathbb{R}^+\}$$

is called a *stochastic process*. For any stochastic process that contains a spatial variable, it is called a *random field*. A formal measure theoretic definition can be found in Geometry of Random Fields by Adler (1980) and Introduction to the Theory of Random Processes by Gikhman and Skorokhod (1969). Functional magnetic resonance images (fMRI) is one example spatiotemporal modeling is necessary.

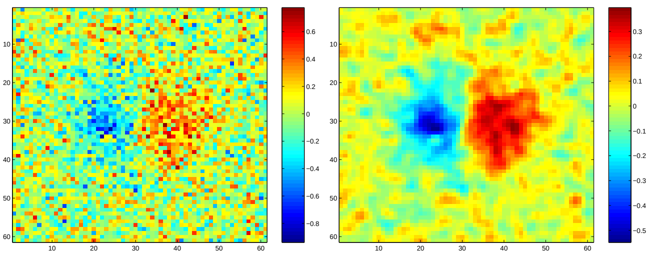

Figure 1: The first image is the realization of white noise which would be unrealistic for modelling room temperature due to discontinuity. The second image is the realization of a Gaussian field which show realistic continuous measurements.

2. *Gaussian random fields.* A random vector $X = (X_1, \cdots, X_m)$ is multivariate normal if $\sum_i c_i X_i$ is Gaussian for every possible choice of c_i.

   A random fields $\epsilon(x) \in \mathbb{R}^N$ is a *Gaussian random field* if $\epsilon(x_1), \cdots, \epsilon(x_m)$ is multivariate normal for any $x_i \in M \subset \mathbb{R}^N$. ϵ is a mean zero Gaussian field if $\mathbb{E}\epsilon(x) = 0$ for all x. The covariance function of mean zero field is defined as

$$R(x,y) = \mathbb{E}\epsilon(x)\epsilon(y).$$

   The variance of field ϵ at fixed point x is $R(x,x)$. Mean zero Gaussian field is completely characterized by the covariance function. A Gaussian random vector field is defined similarly. $e(x) = (e_1(x), \cdots, e_n(x))'$ is a Gaussian vector field if e_i are Gaussian fields. We may generalize it further to matrix fields and tensor fields. Diffusion tensor images (DTI) can be modelled as a tensor field. Unfortunately, it will not be Gaussian tensor fields although it is possible to make images more Gaussian via kernel smoothing.

3. *Independence.* Two fields e_1 and e_2 are independent if $e_1(x)$ and $e_2(y)$ are independent for every x and y. For mean zero Gaussian fields, e_1 and e_2 are independent if and only if the cross-covariance function

$$R(x,y) = \mathbb{E}e_1(x)e_2(y) = 0$$

   for all x and y.

   Problem 1. For given two arbitrary mean zero Gaussian fields, is there mapping that makes them independent?

4. *Infinite-dimensional vector space.* Let $\mathcal{L}$ be a collection of random fields. Suppose $e_1, e_2 \in \mathcal{L}$. $\mathcal{L}$ forms a *vector space* if $c_1e_1 + c_2e_2 \in \mathcal{L}$ for any c_1 and c_2. Obviously the collection of Gaussian fields form a vector space. Since there is no way to represent every element in $\mathcal{L}$ with finite basis fields, it is an infinite-dimensional vector space. However, there exists a finite vector space L_p that is the closest to $\mathcal{L}$ in the least-squares sense (Hilbert space theory). Real and Complex Analysis by Rudin (1986) gives a very nice undergraduate level introduction of Hilbert space. Functional Analysis by Rudin (1991) and A Course in Functional Analysis by Conway (1997) gives concise graduate level treatment of the subject matter.

   For any linear operator f, $f(\mathcal{L}) \subset \mathcal{L}$. It can be shown that differentiation of Gaussian fields is again Gaussian in the mean-square sense.