

Stat 992: Lecture 12

Karhunen-Loève Expansion II.

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1. *Global inference.* As one application of Karhunen-Loève expansion, we show how we do a simple inference on random fields. Assuming simple additive model

$$Y(x) = \mu(x) + \epsilon(x), x \in \Omega$$

where μ is true unknown signal while ϵ is a zero mean Gaussian field with isotropic covariance function R . We are interested in seeing if

$$H_0 : \mu(x) = 0 \text{ for all } x \in \Omega.$$

The alternate hypothesis is

$$H_1 : \mu(x) \neq 0 \text{ for some } x \in \Omega.$$

Assuming the continuity of signal μ , the null hypothesis is equivalent to the following hypothesis:

$$H'_0 : \int_{\Omega} \mu^2(x) dx = 0.$$

So under H_0 , our simple test statistic is

$$\int_{\Omega} Y^2(x) dx = \int_{\Omega} \epsilon^2(x) dx.$$

We can come up with different test statistics but their distribution might be very hard to compute analytically.

From Karhunen-Loève expansion, we have

$$\epsilon(x) = \sum_{j=0}^{\infty} \epsilon_j \phi_j(x)$$

with independent $\epsilon_j \sim N(0, \sigma_j^2)$ and σ_j^2 and ϕ_j are i -th eigensystem for covariance function R with respect to domain Ω . Then

$$\begin{aligned} \int_{\Omega} \epsilon^2(x) dx &= \int_{\Omega} \left[\sum_j \epsilon_j \phi_j(x) \right]^2 dx \\ &= \int_{\Omega} \sum_{i,j} \epsilon_i \epsilon_j \phi_i(x) \phi_j(x) dx \\ &= \sum_{i,j} \epsilon_i \epsilon_j \int_{\Omega} \phi_i \phi_j dx \\ &= \sum_{j=0}^{\infty} \epsilon_j^2 \end{aligned}$$

This is the infinite sum of i.i.d. scaled χ^2 random variable with different degrees of freedom. There is no close form of the distribution so we will approximate the distribution with a single scaled χ^2 distribution by matching the moments. See F.E. Satterthwaite, An approximate distribution of estimates of variance components, Biometrics 2:110-114, 1946 for details. Approximately

$$\sum_{j=0}^{\infty} \epsilon_j^2 \stackrel{D}{\sim} c \chi_{\nu}.$$

Note that

$$\mathbb{E} \sum_{j=0}^{\infty} \epsilon_j^2 = \sum_{j=0}^{\infty} \sigma_j^2 < \infty$$

and

$$\text{Var} \sum_{j=0}^{\infty} \epsilon_j^2 = \sum_{j=0}^{\infty} \text{Var} \epsilon_j^2 = 2 \sum_{j=0}^{\infty} \sigma_j^4.$$

To see this we compute the higher order moment of Gaussian random variable $Z_i \sim N(0, \sigma^2)$ via decomposition

$$\begin{aligned} \mathbb{E}[Z_1 Z_2 Z_3 Z_4] &= \mathbb{E}[Z_1 Z_2] \mathbb{E}[Z_3 Z_4] + \mathbb{E}[Z_1 Z_3] \mathbb{E}[Z_2 Z_4] \\ &\quad + \mathbb{E}[Z_1 Z_4] \mathbb{E}[Z_2 Z_3]. \end{aligned}$$

Now let $Z_1 = Z_2 = Z_3 = Z_4$. Then $\mathbb{E} Z_j^4 = 3\sigma^4$. The corresponding mean and variance of $c\chi_{\nu}$ are $c\nu$ and $2c^2\nu$. Matching the mean and variance, we have $c\nu = \sum \sigma_j^2$ and $c^2\nu = \sum \sigma_j^4$. Solving equations, $c = \sum \sigma_j^4 / \sum \sigma_j^2$ and $\nu = (\sum \sigma_j^2)^2 / \sum \sigma_j^4$. Based on this approximate distribution, the inference can be performed¹

Problem 23. Suppose we have a model for a vector field $Y(x) = \mu(x) + \epsilon(x), x \in \Omega$ where ϵ is

¹As pointed out by Prof. Tsui, the higher order moment of zero mean Gaussian random variable can be performed by differentiating the moment generating function $e^{-t^2/2}$ which yields the Hermite polynomials.

a n -dimensional zero mean Gaussian vector field.
Perform hypothesis testing on null hypothesis H_0 :
 $\mu(x) = 0$ for all $x \in \Omega$.

Homework 3. Solve Problems 13-23 by March 1
11:00am. Solve as many problems as you can.