

Stat 992: Lecture 14

Kernel Smoothing III.

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Theorem. If $Z \sim N(0, 1)$,

$$\mathbb{E}Z^{2m} = 1 \cdot 3 \cdot 5 \cdots (2m - 1).$$

Proof. (Prof. Tsui) For density $f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$, $f'(x) = -xf(x)$. From integration by parts

$$\begin{aligned} \int x^{2m} f(x) dx &= - \int_{-\infty}^{\infty} x^{2m-1} f'(x) dx \\ &= (2m - 1) \int x^{2(m-1)} f(x) dx. \end{aligned}$$

Hence we have recurrence

$$\mathbb{E}Z^{2m} = (2m - 1)\mathbb{E}Z^{2(m-1)}.$$

Solve this with initial condition $\mathbb{E}Z^0 = 1$. $\square$

Problem 6. Is the variance of signal smaller after smoothing? Come up with interesting inequality involving variances. Note that $\text{Var}\hat{\mu}(t) = R(t, t)$.

Partial solution. For stochastic signal $Y(x) = \mu(x) + \epsilon(x)$. We showed that the covariance function of $K_\sigma * Y$ is

$$\begin{aligned} R(x, y) &= \int \int K_\sigma(x - x') K_\sigma(y - y') R_Y(x', y') dx' dy' \\ &= K_\sigma *_x K_\sigma *_y R_Y(x, y) \end{aligned}$$

where $*_x$ and $*_y$ indicate convolution with respect to argument x and y respectively. Let $x = y$ and we have

$$\text{Var}[K_\sigma * Y(x)] = K_{\sqrt{2}\sigma} * \text{Var}Y(x).$$

Consider heat equation

$$\frac{1}{\sigma} \frac{\partial f}{\partial \sigma} = \Delta f$$

for nonnegative function $f \geq 0$. Suppose x_0 is the point that gives local minimum of $f(x, \sigma)$ at fixed σ . Then it is trivial to show that $\Delta f(x_0, \sigma) < 0$. Hence $\frac{\partial f}{\partial \sigma} < 0$. Hence solution to the heat equation is decreasing at x_0 with increasing σ . Since the solution of heat equation is also a Gaussian kernel smoother,

$$K_{\sqrt{2}\sigma} * \text{Var}Y(x_0) < \text{Var}Y(x_0)$$

where x_0 is now a point that gives local maximum of $\text{Var}Y(x)$.

Problem 8. Using the Fourier transform and its inverse transform, implement 2D Gaussian kernel smoothing in computer. In MATLAB, you can perform finite fourier transform.

Solution. (Partial solution given by Shijie Tang) We define Fourier integral transform $\mathcal{F}$ such that

$$\mathcal{F}f(w) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle w, x \rangle} dx$$

and its inverse

$$\mathcal{F}^{-1}f(x) = \int_{\mathbb{R}^n} f(w) e^{2\pi i \langle x, w \rangle} dw.$$

See Folland's Real Analysis for details. For $f, g \in L(\mathbb{R})$, $\mathcal{F}^{-1}(\mathcal{F}f) = \mathcal{F}(\mathcal{F}^{-1}f) = f$ almost everywhere. Then it can be shown that

$$\mathcal{F}(f * g) = \mathcal{F}f \cdot \mathcal{F}g.$$

Hence

$$K_\sigma * Y = \mathcal{F}^{-1}[\mathcal{F}K_\sigma \cdot \mathcal{F}Y]. \quad (1)$$

Gaussian kernel of the form $K(x) = e^{-\pi \|x\|^2}$ is invariant under the Fourier transform. Then $K_\sigma(x) = \frac{1}{(2\pi)^{n/2} \sigma^n} K(\frac{x}{\sqrt{2\pi}\sigma})$. It can be shown that for an invertible linear transform T

$$\mathcal{F}(f \circ T) = |\det T|^{-1} \mathcal{F}f \circ T^{-1}.$$

So

$$\mathcal{F}K_\sigma(w) = K(\sqrt{2\pi}\sigma w) \quad (2)$$

In MATLAB, we implement Gaussian kernel smoothing using discrete fourier transform FFT2 and its inverse IFFT2 which are implemented as the Riemann sum of the integral operators. There is a way to speed things up using the fast fourier transform.