

Stat 992: Lecture 16

Diffusion Smoothing II.

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1. *Discrete maximum principle.* Since the diffusion smoothing and kernel smoothing are equivalent, The diffused signal $f(x_i, t_{j+1})$ must be bounded by the minimum and the maximum of signal. So

$$\begin{aligned} f(x_i, t_{j+1}) &= f(x, t_j) + \delta t \frac{\partial^2 f}{\partial x^2}(x, t_j) \\ &\leq \max [f(x_{i-1}, t_j), f(x_i, t_j), f(x_{i+1}, t_j)] \end{aligned}$$

and also we can bound it by below. Hence

$$\delta t \leq \max \left[\left| \frac{f(x_{i-1}, t_j) - f(x_i, t_j)}{\frac{\partial^2 f}{\partial x^2}} \right|, \left| \frac{f(x_{i+1}, t_j) - f(x_i, t_j)}{\frac{\partial^2 f}{\partial x^2}} \right| \right]$$

2. *Iterated weighted averaging* The idea of smoothing data by directly solving either ordinary differential equation (ODE) or partial differential equation (PDE) originates from signal and image processing. In the previous lecture, the finite difference is given by

$$f(x_i, t_{j+1}) = \lambda f(x_{i-1}, t_j) + (1 - 2\lambda) f(x_i, t_j) + \lambda f(x_{i+1}, t_j)$$

where $\lambda = \delta t / \delta x^2$. Hence diffusion smoothing can be considered as an iterative weighted averaging. Obviously $0 \leq \lambda \leq 1/3$. Depending on the value of λ , the property of smoothing changes. When $\lambda = 0$, we are not smoothing while when $\lambda = 1/3$, we are assigning equal weights to three neighboring observations.

3. *Random walk.* Suppose there is a particle at x_i at time t_j . Suppose we have a symmetric random walk such that the probability of jumping either left or right is r and the probability of staying is $1 - r$. Interpret $f(x_i, t_j)$ as the probability of the particle staying at position x_i at time t_j .

Problem 26. Assuming we have grid system $x_i = i\delta x, i \in \mathbb{Z}$ and particle was at rest at $x_0 = 0$ at time $t_0 = 0$. Suppose the transition probability of random walk is given by

$$P(x_{i-1}|x_i) = r, P(x_i|x_i) = 1-2r, P(x_{i+1}|x_i) = r.$$

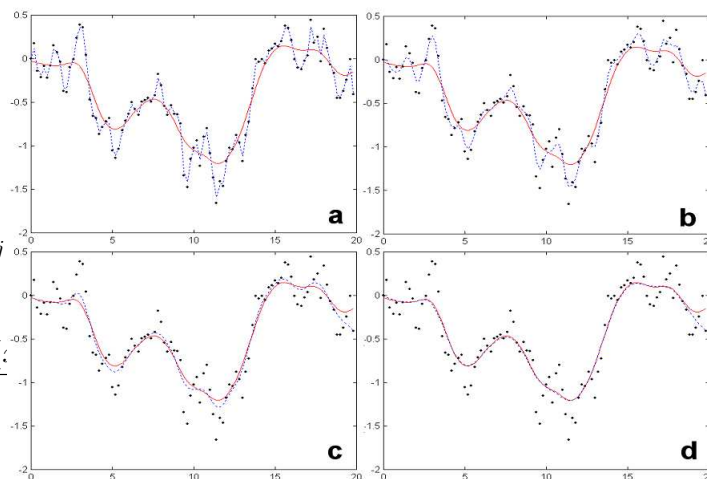


Figure 1: It is shown that the diffusion smoothing (dotted lines) converges to kernel smoothing as the number of iterations increase (5, 25 and 50 iterations respectively).

Find the probability of finding particle at x_m at time t_p . What happens when $\delta x \rightarrow 0$?

4. *Bivariate diffusion smoothing.* Solving diffusion equation in $\mathbb{Z}^2$ depends on estimating 2-dimensional Laplacian. Using the finite difference scheme twice, we have

$$\Delta f(x, t) = f(x + \delta x, y, t) + f(x, y + \delta y, t) + f(x - \delta x, y, t) + f(x, y - \delta y, t) - 4f(x, y, t).$$

This uses 4 closest neighbors of pixel position (x, y) to set up an iteration. It is also possible to incorporate 4 corners $(x \pm \delta x, y \pm \delta y)$.

Problem 27. Estimate 2-dimensional Laplacian using 8 pixel positions around (x, y) so that the Laplacian at (x, y) is expressed as the weighted average of nine terms. Based on both 4 and 8 neighbor Laplacian, implement diffusion smoothing in MATLAB with $t = 10$ and $t = 20$ and compare your results. For data, use `sagittal.data`. Which Laplacian gives faster convergence and why?