

Stat 992: Lecture 20

Bivariate smoothing on a sphere

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1. **Problem 15.** Find spherical Laplacian.

Solution. (Shubing Wang). Parametrize unit sphere S^2 by spherical coordinates. Let $p = (x, y, z) \in S^2$.

$$x = \sin \theta \cos \psi, \quad y = \sin \theta \sin \psi, \quad z = \cos \theta.$$

There are slightly different parameterizations but this is the usual standard. The tangent vectors which form basis in the tangent space T_p is

$$\frac{\partial p}{\partial \theta} = (\cos \theta \cos \psi, \cos \theta \sin \psi, -\sin \theta)$$

and

$$\frac{\partial p}{\partial \psi} = (-\sin \theta \sin \psi, \sin \theta \cos \psi, 0)$$

Note that

$$\left\langle \frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial \psi} \right\rangle = 0, \quad \left\langle \frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial \theta} \right\rangle = 1, \quad \left\langle \frac{\partial p}{\partial \psi}, \frac{\partial p}{\partial \psi} \right\rangle = \sin^2 \theta.$$

So we have orthogonal coordinates and this is why you can construct orthogonal grid system on a sphere. Then the Riemannian metric tensors are

$$g_{11} = 1, g_{12} = g_{21} = 0, g_{22} = \sin^2 \theta.$$

Note that $\det g = \sin^2 \theta$ and $g^{11} = 1, g^{12} = g^{21} = 0, g^{22} = 1/\sin^2 \theta$. From previous lectures, the Laplace-Beltrami operator on manifolds is given by

$$\Delta = \frac{1}{\det G^{1/2}} \sum_{i,j=1}^2 \frac{\partial}{\partial u^i} \left(\det G^{1/2} g^{ij} \frac{\partial}{\partial u^j} \right)$$

where $(u^1, u^2) = (\theta, \psi)$. Because of the orthogonality, terms cancel out and we have

$$\begin{aligned} \Delta_{S^2} &= \frac{1}{\sin \theta} \left[\frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{\partial}{\partial \psi} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \psi} \right) \right] \\ &= \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \psi^2} \end{aligned}$$

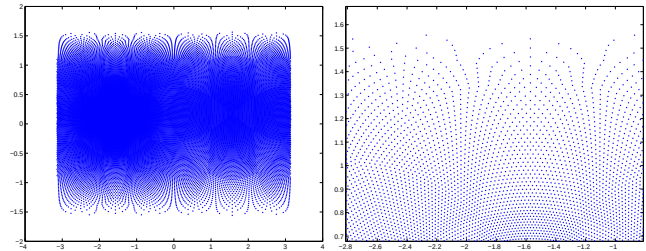


Figure 1: Left: Coordinates (θ_i, ψ_i) where data are observed. Sampling density is lower in polar regions. Right: North pole regions are magnified to show the non-conformity of sampling density. If one tries to formulate nonparametric regression in these nonregular data points, one encounters numerical implementation problem.

2. *Diffusion smoothing on a sphere.* Based on the spherical Laplacian given above, we will construct bivariate diffusion smoothing in S^2 . Suppose we have observations $Y(x), x \in S^2$. Then we need to solve

$$\frac{\partial f}{\partial t} = \Delta_{S^2} f$$

with the initial condition $f(\theta, \psi, 0) = Y(\theta, \psi)$.

Suppose we have polygonal mesh data available. Assume $p_1, \dots, p_m$ to be vertices are around p_0 . The corresponding spherical coordinates of vertex $p_i = (x_i, y_i)$ is denoted by (θ_i, ψ_i) . Then following the idea of estimating Laplacian in polygonal surfaces, we fit a quadratic polynomials locally via the least-squares estimation to estimate derivatives $\frac{\partial}{\partial \theta}, \frac{\partial^2}{\partial \theta^2}, \frac{\partial^2}{\partial \psi^2}$. However, the problem with the spherical coordinates is that near the both north and south poles $\theta = 0, \pi$, the number of vertices of polygonal surfaces decreases. So when we flatten the data defined on a sphere to a rectangle

$$R = \{0 \leq \theta \leq \pi, 0 \leq \psi < 2\pi\}$$

there is the problem of lower sampling density on both top and bottom of the rectangular domain. Thus the local polynomial fitting may per-

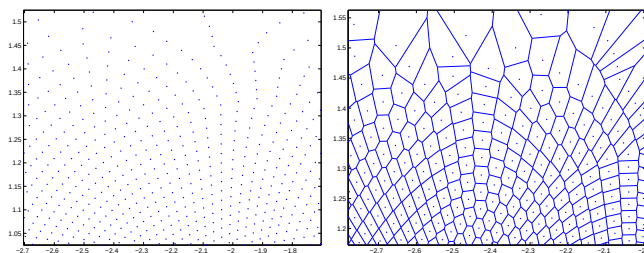


Figure 2: Left: Polar regions are magnified to show the nonuniformity of sampling density. It is numerically nontrivial to implement most of nonparametric regressions if data are spaced nonregularly in higher dimensions. Right: Voronoi diagram. The dual of Voronoi is the Delaunay triangulation.

form badly in these regions. To avoid this parameterization problem, estimation of the Laplacian near the polar regions are done by rotating the sphere by 90 degree, i.e. by adding θ by $\pi/2$. Note that the Laplacian should be invariant under such coordinate transformation. Flattening polygonal surface of S^2 to rectangle R destroys the topology so we need to construct a new vertex connectivity map.

Download data from

<http://tezpur.keck.waisman.wisc.edu/chung/teaching/data/MNImesh/>

```
[tri, coord, nbr, n] = MNIGetmesh('white.obj');
tem = Scoord.^2; for
i = 1:40962
    coord(:, i) = coord(:, i)
                / sqrt(sum(tem(:, i)));
end;
[theta, psi, r] = cart2sph(coord(1, :),
                           coord(2, :), coord(3, :));
```

3. *Delaunay triangulation.* Given n points $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$, triangulation $\mathcal{T}(P)$ of P is a planar subdivision of the convex hull of P into non overlapping triangles with vertices from P . $\mathcal{T}(P)$ is maximal in a sense that adding additional age connecting two vertices in P will make $\mathcal{T}(P)$ non-planar. There are only finitely many different triangulations.

One unique triangulation is given by *Delaunay diagram* $\mathcal{DT}(P)$ which is a dual graph of *Voronoi diagram* $\mathcal{V}(P)$. A Delaunay diagram or Delaunay triangulation is defined as a triangulation of V such that the circumcircle of any triangle in $\mathcal{DT}(P)$ does not contain points of P in its interior. Assuming no four or more points of P are cocircular, it can be shown to be unique. The probability of finite random points forming a circle is zero so most

of Delaunay triangulation encountered in practice should be unique unless the coordinates of data are biased somehow. Algorithms for generating Delaunay diagrams are well known in computational geometry and also available in *MATLAB*. See M. De Berg's Computational Geometry: Algorithms and Applications, 2nd rev. ed. Berlin: Springer-Verlag, pp. 147-163, 2000 and H. Edelsbrunner's Geometry and Topology for Mesh Generation,

```
DT = delaunay(theta, psi);
figure; trimesh(DT, theta, psi, thickness);

voronoi(theta, psi);
```

Problem 30. If n is the number of points in P and k points lie on the boundary of $\mathcal{DT}(P)$, there are A number of triangles and B number of edges in $\mathcal{T}(P)$. Find A and B . Prove your answer and demonstrate for simple examples using built-in functions in *MATLAB*.

Problem 31. Based on Delaunay triangulation, estimate the Laplacian Δ_{S^2} via the least squares estimation and do diffusion smoothing on a sphere. Use `thickness.vv` in the same directory.