

Stat 992: Lecture 22

Brownian motion.

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1. *Spherical harmonics.* By combining the solution from both ODEs, we have the complete set of solutions to the spherical Helmholtz equation:

$$H_n^m(\theta, \psi) = \cos(m\psi)P_n^m(\cos\theta),$$

$$\tilde{H}_n^m(\theta, \psi) = \sin(m\psi)P_n^m(\cos\theta)$$

where $n = 0, 1, 2, \dots, m = 0, 1, \dots, n$. The spherical harmonics satisfy

$$\Delta_{S^2} H_n^m + n(n+1)Y_n^m = \Delta_{S^2} \tilde{H}_n^m + n(n+1)Y_n^m = 0.$$

Note that $\tilde{H}_n^0 = 0$. The corresponding eigenvalues are $\lambda_n = -n(n+1)$. These harmonics are orthogonal but not normalized. It can be shown that

$$\|H_n^0\|^2 = \frac{4\pi}{2n+1}, \|H_n^m\|^2 = \|\tilde{H}_n^m\|^2 = \frac{2\pi(n+m)!}{(2n+1)(n-m)!}.$$

We will assume H_n^m and $\tilde{H}_n^m$ are normalized by dividing by the norms. For simplicity let us order spherical harmonics H_n^m and $\tilde{H}_n^m$ by H_j . Then the spherical harmonics can span any function $f \in L^2(S^2)$ so that

$$f(\theta, \psi) = \sum_{j=0}^{\infty} c_j H_j$$

where $c_j = \langle f, H_j \rangle$.

From S. Rosenberg's *The Laplacian on a Riemannian Manifold* (1997 Cambridge University Press), it can be shown that the spherical heat kernel can be extended as

$$K_t(p, q) = \sum_{j=0}^{\infty} e^{-\lambda_j t} H_j(p) H_j(q).$$

Hence solution to diffusion smoothing on the unit sphere is exactly expressed as

$$f(t, q) = \sum_{j=0}^{\infty} e^{-\lambda_j t} H_j(q) \int_0^{2\pi} \int_0^\pi H_j(p) Y(p) \sin\theta \, d\theta d\psi.$$

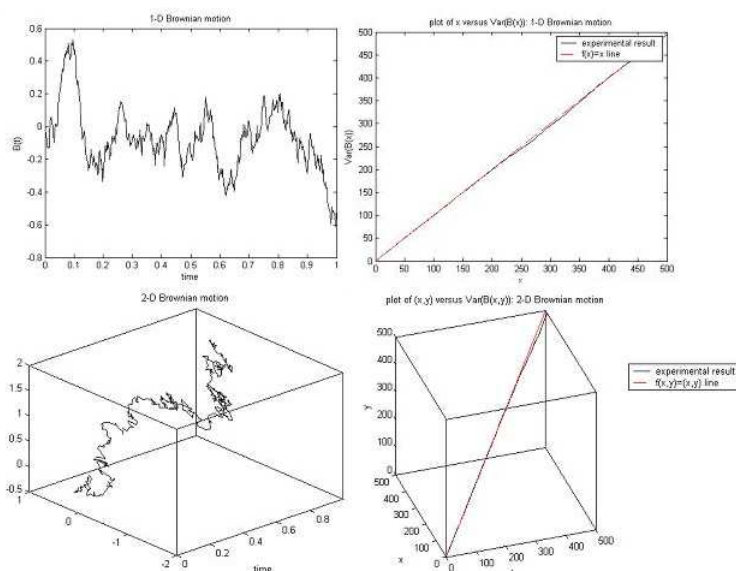


Figure 1: Top: 1D Brownian motion simulation, Bottom: 2D Brownian motion simulation. `plot3` command in MATLAB is used to plot it in 3D.

Problem 18. In MATLAB, simulate 1D and 2D version of Brownian motion based on the above properties and check $\text{Var}B(x) = x$ by simulating 10000 Brownian motions. What is the corresponding 2D version?

Solution. (Tulaya Limpiti). Simulation is based on the property of the independent increment of Brownian motion.

Problem 19. Prove the covariance function of the $W(x) = \frac{dB(x)}{dx}$ is proportional to the Dirac-delta function.

Solution. (Tulaya Limpiti) Let $R_W(x, y) = \mathbb{E}[W(x)W(y)]$. We will show that

$$\begin{aligned} & \int_0^u \int_0^u f(x)f(y)R_W(x, y) \, dx dy \\ &= \int_0^u \int_0^u f(x)f(y)\delta(x-y) \, dx dy \\ &= \int_0^u f^2(y) dy \end{aligned}$$

for all possible smooth test function f . The white noise process W is defined as

$$\int_0^u f(y)W(y) dy = -f(u)B(u) + \int_0^u f'(y)B(y) dy$$

for all possible smooth test function f . Then

$$\begin{aligned} & \int_0^u \int_0^u f(x)f(y)R_W(x,y) dx dy \\ &= \mathbb{E} \int_0^u \int_0^u f(x)f(y)W(x)W(y) dx dy \\ &= \mathbb{E} \left[\int_0^u f(y)W(y) dy \right]^2 \\ &= \mathbb{E} \left[-f(u)B(u) + \int_0^u f'(y)W(y) dy \right]^2 \\ &= f^2(u)\mathbb{E}B^2(u) - 2f(u) \int_0^u f'(y)\mathbb{E}[B(u)W(y)] dy \\ &\quad + \int_0^u \int_0^u f'(x)f'(y)\mathbb{E}[B(x)B(y)] dx dy \\ &= uf^2(u) - 2f(u) \int_0^u yf'(y) dy \\ &\quad + 2 \int_0^u f'(y) dy \int_0^y xf'(x) dx. \end{aligned}$$

In the last line, we used the fact $\mathbb{E}[B(x)B(y)] = \min(x,y) = x$ if $x \leq y$. Now following Tulaya's argument, differentiate the above expression and get $f^2(u)$. Then taking the integration, we prove the statement. $\square$

Problem 21. Let X be a Brownian motion with covariance function given by $R(x,y) = \min(x,y)$. Find the eigenvalues and eigenfunctions. Based on this eigensystem, expand the Brownian motion via the Karhunen-Loève expansion. Based on the Karhunen-Loève expansion, simulate Brownian motion in MATLAB.

Solution. (Tulaya Limpiti). This is a standard example in the Karhunen-Loève expansion. For simplicity assume that the Brownian motion is defined on $[0, 1]$. We need to solve

$$\begin{aligned} \lambda\phi(x) &= \int_0^u R(x,y)\phi(y) dy \\ &= \int_0^1 \min(x,y)\phi(y) dy \\ &= \int_0^x y\phi(y) dy + \int_x^1 x\phi(y) dy \end{aligned}$$

Differentiating both sides with respect to x , we get

$$\begin{aligned} \lambda\phi'(x) &= \int_x^1 \phi(y) dy, \\ \lambda\phi''(x) &= -\phi(x). \end{aligned}$$

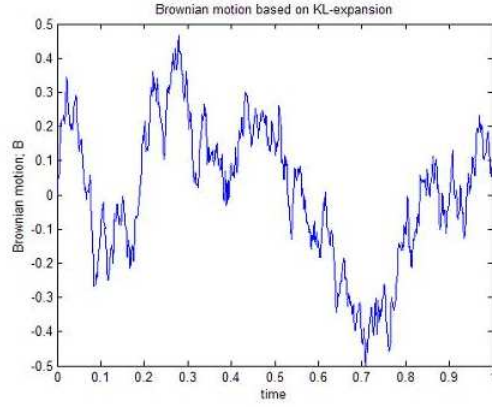


Figure 2: Top: 1D Brownian motion simulation based on the Karhunen-Loève expansion.

Two independent solutions are $\sin \frac{x}{\sqrt{\lambda}}$, $\cos \frac{x}{\sqrt{\lambda}}$. From the boundary condition $\phi(0) = 0$, $\phi(x) = \sin \frac{x}{\sqrt{\lambda}}$. From $\phi'(1) = 0$, we must have eigenvalues

$$\lambda_j = \left[\frac{2}{(2j+1)\pi} \right]^2.$$

Note that $\int_0^1 \phi^2(x) dx = \frac{1}{2}$. So normalizing the eigenfunctions, we have $\phi_j(x) = \sqrt{2} \sin \frac{x}{\sqrt{\lambda_j}}$. From the Karhunen-Loève expansion, Brownian motion is written as

$$B(x) = \sum_{j=0}^{\infty} Z_j \psi_j(x)$$

where i.i.d. $Z_j \sim N(0, \lambda_j)$.

Problem 34. When you smooth Brownian motion with the covariance function $R(x,y) = \min(x,y)$ via kernel smoothing with K_σ , what do you get?