

Stat 992: Lecture 29

Finite element methods II.

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1. *Barycentric coordinates.* Parameterization gives an automatic way of estimating the Laplacian and in turn it can be used to set up diffusion smoothing. However, in a very complex manifold, simple parameterization may not be given. In that situation, we show how to solve a diffusion equation using the finite element method (FEM). Let N_T be the number of triangles in the triangular mesh S that is the discrete estimation of true surface $\partial\Omega$. We seek an approximate solution F_i in triangle T_i such that the solution $F_i(x, t)$ is continuous across neighboring triangles. The approximate solution F for the whole region is then

$$F(x, t) \doteq \sum_{i=1}^{N_T} F_i(x, t).$$

A slightly different formulation of FEM for the surface flattening problem is given in Angenent *et al.*. Let p_{i1}, p_{i2}, p_{i3} be the vertices of element T_i . In T_i , we interpolate F_i linearly by

$$F_i(x, t) = \sum_{j=1}^3 \xi_{ij}(x) F(p_{ij}, t),$$

where nonnegative ξ_{ik} are given by the *barycentric coordinates*. In the barycentric coordinates, any point $x \in T_i$ is uniquely determined by two conditions:

$$x = \sum_{k=1}^3 \xi_{ik}(x) p_{ik}, \quad \sum_{k=1}^3 \xi_{ik}(x) = 1.$$

If $x \in T_i$, we let $\xi_{ik} = 0$. Note that $0 \leq \xi_{ik} \leq 1$. Let G be an arbitrary piecewise linear function on S . Then it can be uniquely written as

$$G(x) = \sum_{i=1}^{N_T} \xi_{i1}(x) G_{i1} + \xi_{i2}(x) G_{i2} + \xi_{i3}(x) G_{i3},$$

where $G_{ik} = G(p_{ik})$ are the values of function G evaluated at vertex p_{ik} of T_i . Another piecewise linear function $F(x, t)$ is given similarly so that $F_{ik} = F(p_{ik}, t)$.

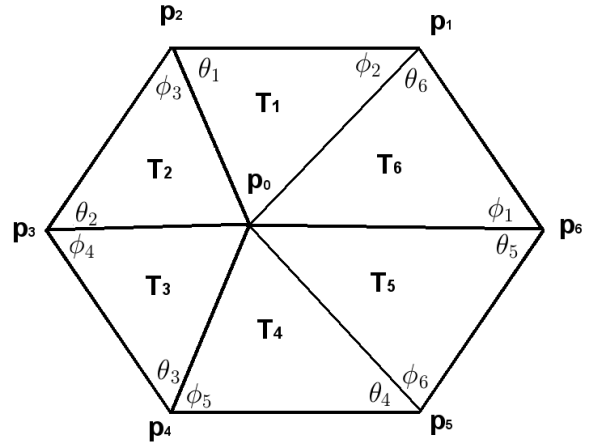


Figure 1: A typical triangular elements.

2. *Diffusion equation in an element.* The integral version of diffusion equation $\partial_t F = \Delta F$ can be written as

$$\int_{T_i} G \partial_t F \, dT = - \int_{T_i} \langle \nabla F, \nabla G \rangle \, dT. \quad (1)$$

The left-hand term in (1) is

$$\begin{aligned} \int_{T_i} G \partial_t F \, dT &= \sum_{k,l=1}^3 G_{ik} \partial_t F(p_{il}, t) \int_{T_i} \xi_{ik} \xi_{il} \, dT \\ &= [G_i]' [A^i] \frac{d}{dt} [F_i], \end{aligned}$$

where $[G_i] = (G_{i1}, G_{i2}, G_{i3})'$, $[F_i] = (F_{i1}, F_{i2}, F_{i3})'$

$$[A^i] = \frac{|T_i|}{12} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \quad (2)$$

where $|T_i|$ is the area of the triangular element T_i .

Similarly the right-hand term in (1) can be written as

$$\begin{aligned} \int_{T_i} \langle \nabla F, \nabla G \rangle \, dT &= \sum_{k,l=1}^3 G_{ik} F(p_{il}, t) \int_{T_i} \langle \nabla \xi_{ik}, \nabla \xi_{il} \rangle \, dT \\ &= [G_i]' [C^i] [F_i], \end{aligned}$$

$$\begin{pmatrix} C_{00}^1 + \dots + C_{00}^6 & C_{01}^1 + C_{01}^6 & C_{02}^1 + C_{02}^2 & C_{03}^2 + C_{03}^3 & C_{04}^3 + C_{04}^4 & C_{05}^4 + C_{05}^5 & C_{06}^5 + C_{06}^6 \\ C_{01}^1 + C_{01}^6 & C_{11}^1 + C_{11}^6 & C_{12}^1 & 0 & 0 & 0 & C_{16}^6 \\ C_{02}^1 + C_{02}^2 & C_{12}^1 & C_{22}^1 + C_{22}^2 & C_{23}^2 & 0 & 0 & 0 \\ C_{03}^2 + C_{03}^3 & 0 & C_{23}^2 & C_{33}^2 + C_{33}^3 & C_{34}^3 & 0 & 0 \\ C_{04}^3 + C_{04}^4 & 0 & 0 & C_{34}^3 & C_{44}^3 + C_{44}^4 & C_{45}^4 & 0 \\ C_{05}^4 + C_{05}^5 & 0 & 0 & 0 & C_{45}^4 & C_{55}^4 + C_{55}^5 & C_{56}^5 \\ C_{06}^5 + C_{06}^6 & C_{16}^6 & 0 & 0 & 0 & C_{56}^5 & C_{66}^5 + C_{66}^6 \end{pmatrix}.$$

Figure 2: Global coefficient matrix.

where $[C^i] = (C_{kl}^i)$, $C_{kl}^i = \int_{T_i} \langle \nabla \xi_{i_k}, \nabla \xi_{i_l} \rangle dT$. Since T_i is planar, the gradient $\nabla \xi_{i_k}$ becomes the standard planar gradient. Following Sadiku, it can be shown that $2[C^i]$ is the matrix of the following form:

$$\begin{pmatrix} \cot \theta_{i_2} + \cot \theta_{i_3} & -\cot \theta_{i_3} & -\cot \theta_{i_2} \\ -\cot \theta_{i_3} & \cot \theta_{i_1} + \cot \theta_{i_3} & -\cot \theta_{i_1} \\ -\cot \theta_{i_2} & -\cot \theta_{i_1} & \cot \theta_{i_1} + \cot \theta_{i_2} \end{pmatrix},$$

where θ_{i_k} is the interior angle of vertex p_{i_k} . Equating the left and right hand sides, we get

$$[G_i]'[A^i] \frac{d}{dt}[F_i] = -[G_i]'[G^i][F_i].$$

Since the equation (3) should be satisfied for an arbitrary vector $[G_i]$, we have a system of ordinary differential equations (ODE) given by

$$\frac{d[F_i]}{dt} = -[A^i]^{-1}[C^i][F_i] \text{ for all } i.$$

This is the diffusion equation on planer element T_i .

3. *Assembling elements* Having discretized an element, the next step is to assemble all such elements in m incident triangles around vertex p . Let $p_1, \dots, p_m$ be the m neighboring vertices around $p = p_0$ in the counter-clockwise direction. Let p, p_i, p_{i+1} be the vertices of the element T_i (Figure 1). Then from combining elements, we have

$$\begin{aligned} \int_{T_1 \cup \dots \cup T_m} \langle \nabla F, \nabla G \rangle dT &= \sum_{i=1}^m \int_{T_i} \langle \nabla F, \nabla G \rangle dT \\ &= [G]'[C][F], \end{aligned}$$

where $[F] = [F(p, t), F(p_1, t), \dots, F(p_m, t)]'$ and $[G] = [G(p), G(p_1), \dots, G(p_m)]'$. The matrix $[C] = (C_{ij})$ is called the *global coefficient matrix*, which is the assemblage of individual element coefficients. The contribution to C_{ij} comes from all elements containing vertices i and j . In the case of a hexagonal triangulation in Figure 1, $[C]$ is given in Figure 2. Similarly

$$\int_{T_1 \cup \dots \cup T_m} G \partial_t F dT = [G]'[A] \frac{d[F]}{dt}, \quad (3)$$

where $[A] = (A_{ij})$ has the same structure as $[C]$, i.e. write $A_{01} = A_{01}^1 + A_{01}^6$ instead of $C_{01} = C_{01}^1 + C_{01}^6$. Equating the above equations we have

$$[G]'[A] \frac{d[F]}{dt} = -[G]'[C][F].$$

Since equation (4) should be satisfied for an arbitrary piecewise linear function G , we have a discrete diffusion equation on m elements $T_1, \dots, T_m$ given by

$$\frac{d[F]}{dt} = -[A]^{-1}[C][F].$$

At first glance, it seems like we need to solve a system of linear equations iteratively. Note that the first row of the simultaneous ODE (4) gives the discrete diffusion equation at the vertex $p = p_0$:

$$\frac{dF(p, t)}{dt} = - \sum_{i,k=0}^m A_{0k}^{-1} C_{ki} F(p_i, t),$$

where A_{0k}^{-1} is the $0k$ -th element of A^{-1} . Comparing this with the diffusion equation $\partial_t F(p, t) = \Delta F(p, t)$, we can see that the right-hand side of equation (4) is the discrete estimation of Laplacian of function F evaluated at vertex p . Simplifying the matrix computation using the computational algebraic system MAPLE, we have the FEM estimation for the Laplace-Beltrami operator given by

$$\hat{\Delta} F(p) = \sum_{i=1}^m w_i (F(p_i) - F(p))$$

with the weights $w_i = (\cot \theta_i + \cot \phi_i)/|T|$, where θ_i and ϕ_i are the two angles opposite to the edge $p_i - p$ and $|T| = \sum_{i=1}^m |T_i|$ is the sum of the areas of the incident triangles (Figure 1).

Read M.N.O. Sadiku, Numerical Techniques in Electromagnetics. Possibly the easiest introductory book on FEM.

Problem 43. Using only 6 elements in Figure 1 minimize

$$\sum_{i=0}^6 |F(p_i) - f(p_i)|^2 + \lambda \int_{T_1 \cup \dots \cup T_6} \Delta F(p) f(p) d\mu(p)$$

for given continuous functional data f and smoothing bandwidth λ using FEM. Hint: Assume F to be a piecewise linear function and find values $F(p_i)$.