

Stat 992: Lecture 36

Smoothness of random fields

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Solution to Problem 37. (Tulaya Limpiti). The smoothness of noise can be measured via the covariance matrix of the derivative field.

1. Consider zero mean unit variance Gaussian field $Y = K_\sigma * W$ where W Gaussian white noise with the Dirac-delta covariance function. The covariance of Y has been shown to be

$$R_Y(x, y) = \int K_\sigma(x - z) K_\sigma(y - z) dz$$

which should be isotropic, i.e. $R_Y(x, y) = f(\|x - y\|^2)$ for some function f . Field Y does not give unit variance. So we normalize Y by the square root of

$$\text{Var} Y = R_Y(x, x) = f(0) = \int K_\sigma^2(x - z) dz.$$

Note $K_\sigma^2(x) = \frac{1}{(2\sqrt{\pi}\sigma)^n} K_{\sigma/\sqrt{2}}(x)$. So $Y = (2\sqrt{\pi}\sigma)^{n/2} K_\sigma * W$ gives a unit variance field. The way to generate this field numerically is we first simulate white noise $N(0, 1)$ and perform kernel smoothing followed by the multiplication of $(2\sqrt{\pi}\sigma)^{n/2}$.

2. The cross-covariance of the above zero mean unit variance field is given by

$$\begin{aligned} \mathbb{E}[\partial_{x_i} Y(x) \partial_{y_j} Y(y)] &= \partial_{x_i} \partial_{y_j} \mathbb{E}[Y(x) Y(y)] \quad (1) \\ &= \frac{\partial_{x_i} \partial_{y_j} \int K_\sigma(x - z) K_\sigma(y - z) dz}{\int K_\sigma(z) K_\sigma(z) dz} \quad (2) \end{aligned}$$

Then

$$\text{Var}(\partial Y) = \frac{\int \partial K_\sigma(x) [\partial K_\sigma(x)]^T dx}{\int K_\sigma(x) K_\sigma(x) dx}. \quad (3)$$

3. For any isotropic fields Y , it can be shown that $\partial_{x_i} Y$ and $\partial_{x_j} Y$ are uncorrelated if $i \neq j$. It is a trivial property of the isotropic field. We have cross-covariance

$$\mathbb{E}[\partial_{x_i} Y(x) \partial_{y_j} Y(y)] = \partial_{x_i} \partial_{y_j} f(\|x - y\|^2).$$

Since $f(\|x\|^2) = f(\sum_{i=1}^n x_i^2)$, $\partial_{x_j} f(\|x\|^2) = 2x_j f'(\|x\|^2)$. Then,

$$\partial_{x_i} \partial_{x_j} f(\|x\|^2) = 2\delta_{ij} f'(\|x\|^2) + 4x_i x_j f''(\|x\|^2)$$

Letting $x = y$ in equation (1),

$$\mathbb{E}[\partial_{x_i} Y(x) \partial_{y_j} Y(x)] = \partial_{x_i} \partial_{x_j} f(0) = 2\delta_{ij} f'(0).$$

Hence all partial derivatives are uncorrelated.

A different way to see this is to compute the covariance directly from (3).

$$\begin{aligned} \mathbb{E}[\partial_{x_i} Y(x) \partial_{x_j} Y(x)] &= \frac{\int [\partial_{x_i} K_\sigma(x)] [\partial_{x_j} K_\sigma(x)] dx}{\int K_\sigma^2(x) dx} \\ &= \frac{\int \frac{x_i^2}{\sigma^4} K_\sigma^2(x) dx}{\int K_\sigma^2(x) dx} \\ &= \frac{\int \frac{x_i^2}{\sigma^4} K_{\sigma/\sqrt{2}}(x) dx}{\int K_{\sigma/\sqrt{2}}(x) dx} \\ &= \frac{1}{2\sigma^2} \end{aligned}$$

So we have the following theorem

Theorem. For zero mean unit variance isotropic Gaussian field of the form $Y = K_\sigma * W$,

$$\dot{Y} \sim N(0, \frac{1}{2\sigma^2} I_N).$$

Problem 49. If we have zero mean unit variance Gaussian field of the form $Y = K_H * W$, show $\dot{Y} \sim N(0, (HH')^{-1}/2)$. See Worsley et al. (1992) A Three-Dimensional Statistical Analysis for CBF Activation Studies in Human Brain, Journal of Cerebral Blood Flow and Metabolism **12**:900-918. <http://www.math.mcgill.ca/keith/jcbf/jcbf.pdf> where this formula is given.

Next lecture will continue the discussion of anisotropic smoothing.