

Gaussian Mixture Model by Using EM

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1. *2-component Gaussian Mixture Model.* We consider the case that all the 5 parameters

$$\theta^T = (p, \mu_1, \sigma_1, \mu_2, \sigma_2)$$

are unknown. By using EM algorithm:

- **Estimation Step:** define $\tilde{p}_i = P(y_i \sim f_1 | \Theta)$, then

$$\tilde{p} = \frac{pf_1(y_i | \Theta)}{pf_1(y_i | \Theta) + (1-p)f_2(y_i | \Theta)}.$$

So

$$Q(\Theta, \Theta') = \sum_i \left\{ [\tilde{p}_i (\log p' - \log \sqrt{2\pi} - \log \sigma'_1 - \frac{(Y_i - \mu'_1)^2}{2\sigma_1'^2})] + [(1 - \tilde{p}_i) (\log(1 - p') - \log \sqrt{2\pi} - \log \sigma'_2 - \frac{(Y_i - \mu'_2)^2}{2\sigma_2'^2})] \right\}$$

- **Maximization Step:**

$$p' = \frac{\sum_i \tilde{p}_i}{n}$$

$$\mu'_1 = \frac{\sum_i \tilde{p}_i Y_i}{\sum_i \tilde{p}_i}$$

$$\sigma'_1 = \sqrt{\frac{\sum_i \tilde{p}_i (Y_i - \mu'_1)^2}{\sum_i \tilde{p}_i}}$$

$$\mu'_2 = \frac{\sum_i (1 - \tilde{p}_i) Y_i}{\sum_i (1 - \tilde{p}_i)}$$

$$\sigma'_2 = \sqrt{\frac{\sum_i (1 - \tilde{p}_i) (Y_i - \mu'_2)^2}{\sum_i (1 - \tilde{p}_i)}}$$

2. *M-component Gaussian Mixture Model.* It can be derived similarly by using EM with the Maximization step:

$$p'_j = \frac{\sum_i \tilde{p}_{ji}}{n}, \quad 1 \leq j \leq M$$

$$\mu'_j = \frac{\sum_i \tilde{p}_{ji} Y_i}{\sum_i \tilde{p}_{ji}}, \quad \text{if } 1 \leq j < M$$

$$\sigma'_j = \sqrt{\frac{\sum_i \tilde{p}_{ji} (Y_i - \mu'_j)^2}{\sum_i \tilde{p}_{ji}}}, \quad \text{if } 1 \leq j < M$$

$$\mu'_M = \frac{\sum_i (1 - \sum_{j=1}^{M-1} \tilde{p}_{ji}) Y_i}{\sum_i (1 - \sum_{j=1}^{M-1} \tilde{p}_{ji})}$$

$$\sigma'_M = \sqrt{\frac{\sum_i (1 - \sum_{j=1}^{M-1} \tilde{p}_{ji}) (Y_i - \mu'_M)^2}{\sum_i (1 - \sum_{j=1}^{M-1} \tilde{p}_{ji})}}$$

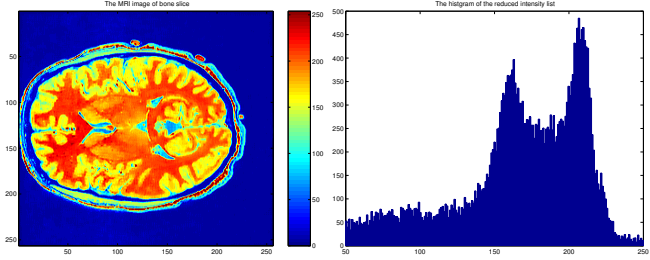


Figure 1: Left: mid sagittal brain image. Right: histogram of image intensity.

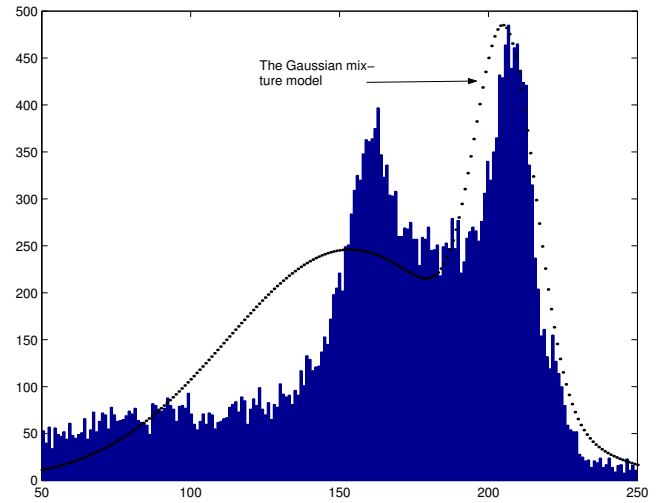


Figure 2: The comparison of the original data and the fitted Gaussian Mixture Model

3. *Segmentation by Using Gaussian Mixture Model.* We start with data as shown in Figure 1. We always should get the histogram as well, since the initial guess is crucial when one is trying to do EM algorithm. A good initial guess will absolutely improve the performance of the programming and histograms provide the clue for good initial guess. One can get the 2-component Gaussian Mixture Model as shown in Figure 2. The result of segmentation as shown in Figure 3.

Intuitively, 3-component Gaussian Mixture Model will give a better result. Figure 4 and Figure 5 show the result of segmentation by using the 3-component Gaussian Mixture Model.

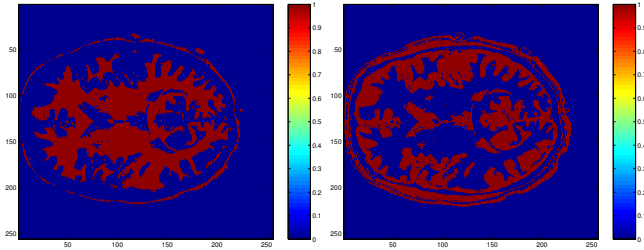


Figure 3: The two components of the 2-component Gaussian Mixture Model

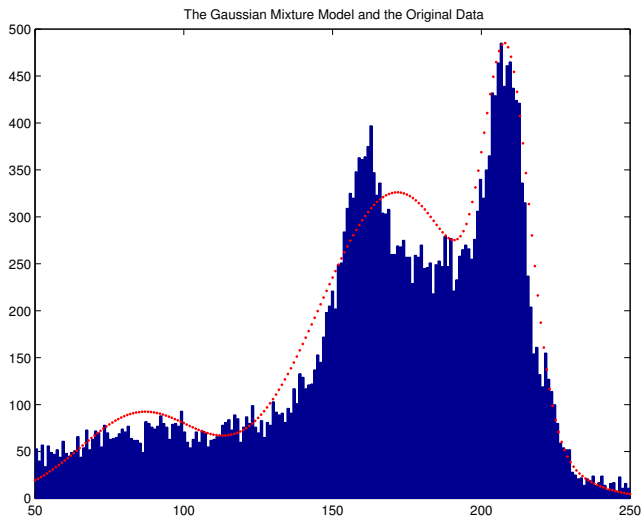


Figure 4: The fitting 3-component Gaussian Mixture Model

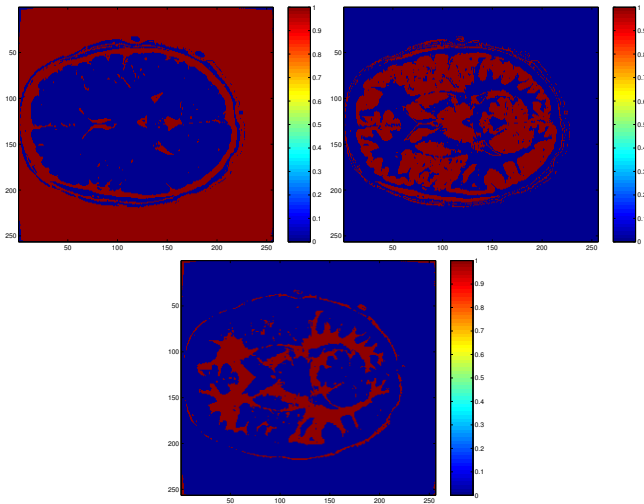


Figure 5: The 3 segments of the 3-component Gaussian Mixture Model