STAT 609 First Exam
11:00am-12:00noon, Oct. 3, 2014

Please show all your work for full credits.

1. Let A, B, C, and D be four events. Suppose that if all B, C, and D occur, then A occurs. Show that $P(A) \geq P(B) + P(C) + P(D) - 2$.

2. A red die and a white die are rolled, where each die has 6 faces numbered $1, ..., 6$.
   (a) Are the events $A = \{4 \text{ on the red die}\}$ and $B = \{\text{sum of dice is odd}\}$ independent?
   (b) Are the events $C = \{5 \text{ on the red die}\}$ and $D = \{\text{sum of dice is } 11\}$ independent?
   (c) Calculate the conditional probability that the sum of dice is larger or equal to 11, given that 5 on the red die.

3. Let X be a random variable having pdf

$$f(x) = \begin{cases} x & 0 < x \leq 1 \\ 2 - x & 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find the cdf of X.
   (b) Obtain $E(X)$.
   (c) If $Y = X^2$, find the pdf of Y.
   (d) Find $E(Y)$, and use the result to find $\text{Var}(X)$.

4. Let X be a discrete random variable with $P(X = a) = p$ and $P(X = b) = 1 - p$, where $0 < p < 1$ and $0 < a < b$. Show that

$$\lim_{n\to\infty} \frac{E(X^{n+1})}{E(X^n)} = b$$

5. Let X be a discrete random variable with pmf

$$f_p(x) = \frac{1}{\log(1/p)} x^{-1}(1-p)^x, \qquad x = 1, 2, ...$$

   where $0 < p < 1$.
   (a) Obtain $E(X)$ by directly calculating

$$E(X) = \sum_{x=1}^{\infty} x f_p(x)$$

   (b) Using the fact that $f_p(x)$ is a pmf for any $p \in (0, 1)$, show that the mgf of X is

$$M_X(t) = \frac{\log(1 - (1-p)e^t)}{\log(p)}, \qquad t < -\log(1-p)$$

   (c) Find $E(X)$ by differentiating $M_X(t)$ and show the result is the same as that in part (a).