STAT 609 Second Exam
11:00am-12:00noon, Nov. 7, 2014

Please show all your work for full credits.

1. Suppose that the covariance matrix of the 2-dimensional random vector (X, Y) is
$$\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$
   (a) (10 points) Find the covariance matrix of $(X - 2Y, 2X - Y)$.
   (b) (10 points) If the joint distribution of (X, Y) is normal, show that it is from an exponential family with the form
$$f_\theta(x, y) = h(x, y)c(\theta)\exp\left(\sum_{i=1}^{k} w_i(\theta)t_i(x, y)\right),$$
   and identify k, the value of the parameter vector θ, and functions h, c, and t_i's.

2. Let P be a random variable having pdf
$$f(p) = \begin{cases} 3p^2 & 0 < p < 1 \\ 0 & \text{otherwise} \end{cases}$$
   Suppose that given $P = p$, X and Y are conditionally independent discrete random variables with
$$P(X = k|P = p) = P(Y = k|P = p) = p(1-p)^k, \qquad k = 0, 1, 2, ...$$
   (a) (10 points) Obtain the pmf of X, i.e., $P(X = k)$, $k = 0, 1, 2, ...$
   (b) (10 points) Obtain the pmf of $Z = \max\{X, Y\}$.
   (c) (10 points) Obtain the conditional pdf of P given $X = x$ and $Y = y$, $x, y = 0, 1, 2, ...$

3. Let X and Y be independent random variables, $X \sim uniform(0, 1)$ and $Y \sim uniform(0, 1)$.
   (a) (10 points) Obtain the joint pdf of $(X + Y, X - Y)$.
   (b) (10 points) Obtain the conditional pdf of $X|(X - Y) = t$.
   (c) (10 points) Obtain the pdf of $|X - Y|$.

4. Let X and Y be independent random variables, $X \sim N(0, 1)$, $Y \sim N(0, 1)$, and
$$Z = \begin{cases} |Y| & X \geq 0 \\ -|Y| & X < 0 \end{cases}$$
   (a) (10 points) Show that $Z \sim N(0, 1)$.
   (b) (5 points) Prove that the joint distribution of (X, Z) is not normal.

5. (5 points) Let X and Y be two random variables having mgf's $M_X(t)$ and $M_Y(s)$ which are finite for $t \in (-h, h)$ and $s \in (-h, h)$ with a positive constant h. Suppose that the mgf of the random vector (X, Y) is
$$M_{(X,Y)}(t, s) = E(e^{tX+sY}) = M_X(t)M_Y(s), \qquad t \in (-h, h),\ s \in (-h, h)$$
   Show that X and Y are independent.