STAT 609 First Exam
2:45-4:45pm, Dec. 17, 2014

Please show all your work for full credits.

1. (8 points) A person has probability 0.005 to have a type of disease. A medical test is positive with probability 0.98 if a person has the disease, and is negative with probability 0.1 if a person does not have the disease. One person had a test and the result was positive. What is the probability that this person actually has the disease?

2. Suppose that the joint distribution of X and Y is normal with covariance matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

   (a) (6 points) Find a value a such that $X - aY$ and $2X + aY$ are independent.

   (b) (6 points) Is the solution a in the previous part unique? Why?

3. (8 points) Let U and V be independent standard random variables, each has $N(0, 1)$ distribution. Find the distribution of $U/(U + V)$.

4. Let X and Y be independent and identically distributed random variables having the $exponential(1)$ pdf.

   (a) (6 points) Obtain the joint pdf of $(X + Y, X - Y)$.

   (b) (6 points) Obtain the conditional pdf of $(X + Y)|(X - Y) = t$.

   (c) (6 points) Obtain the mgf of $X - Y$ and use the mgf to find $E(X - Y)^2$.

5. Let X_n be a random variable having the $Poisson(n)$ distribution, $n = 1, 2, ...$ In the following, the limiting process is with respect to $n \to \infty$.

   (a) (7 points) Show that

$$\frac{X_n}{\sqrt{n}} - \sqrt{n} \text{ converges in distribution to } N(0, 1)$$

   (b) (6 points) For any positive integer k, show that

$$\frac{X_n^k}{n^{k-1/2}} - \sqrt{n} \text{ converges in distribution to } N(0, k^2)$$

   (c) (7 points) Define

$$Y_n = \begin{cases} 1 & X_n = 0 \\ X_n & X_n > 0 \end{cases}$$

   Show that $Y_n - X_n$ converges in probability to 0.

   (d) (6 points) Show that the result in part (b) is still true when k is a negative integer.

(Another problem is on the next page)

6. Let $X_1, ..., X_n$ be a random sample with a pdf

$$f(x) = \begin{cases} e^{-x^2/(2\sigma^2)}/c(\mu, \sigma) & x > \mu \\ 0 & x \le \mu \end{cases}$$

where $\sigma > 0$ and $\mu \in \mathcal{R}$ and $c(\mu, \sigma) = \int_\mu^\infty e^{-t^2/(2\sigma^2)} dt$.

(a) (4 points) Derive a formula of the pdf of the jth order statistic, $1 \le j \le n$. Do not try to simplify integrals of the form $\int_\mu^x e^{-t^2/(2\sigma^2)} dt$.

(b) (8 points) In the case where μ is known and σ is unknown, obtain a complete and sufficient statistic for σ.

(c) (8 points) In the case where μ is unknown and σ is known, obtain a complete and sufficient statistic for μ.

(d) (8 points) Suppose that $\sigma^2 = \mu^2$ is unknown. Obtain a minimal sufficient statistic.