STAT 609 First Exam
1:00pm-2:15pm, Oct. 7, 2015

Please show all your work for full credits.

1. (10 points) Let A and B be events with $0 < P(A) < 1$ and $0 < P(B) < 1$. Suppose that $P(A|B) > P(A|B^c)$, where the superscript c denotes complement. Show that $P(B|A) > P(B)$.

2. (20 points) A urn contains six balls represented by $1, ..., 6$. One ball is to be drawn at random from the urn. Let $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 2, 3, 4\}$, and $D = \{1, 2, 5\}$ be events.

   (a) Are the events A and B independent?

   (b) Are the events A and B conditionally independent given C?

   (c) Are the events D and B independent?

   (d) Are the events D and B conditionally independent given C?

3. (30 points) Let X be a random variable having pdf

$$f(x) = \begin{cases} ax^2 & 0 < x \leq 1 \\ \frac{b}{x^2} & 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   where $a > 0$ and $b > 0$ are constants.

   (a) Find the value of a and b such that $E(X^2) = 1$.

   (b) Find the cdf of X.

   (c) Find the pdf of $Y = (X - 1)^2$.

4. (15 points) Let $x_1, ..., x_n$ be n numbers satisfying $\sum_{i=1}^n x_i = 0$. Prove the following inequalities using the three well-known inequalities in probability theory.

$$\text{(a)} \quad \left(\frac{1}{n} \sum_{i=1}^n |x_i| \right)^2 \leq \frac{1}{n} \sum_{i=1}^n x_i^2$$

$$\text{(b)} \quad \left(\prod_{i=1}^n |x_i| \right)^{1/n} \leq \frac{1}{n} \sum_{i=1}^n |x_i|$$

$$\text{(c)} \quad \frac{\text{number of } |x_i|\text{'s} > \epsilon}{n} \leq \frac{1}{\epsilon^2 n} \sum_{i=1}^n x_i^2 \quad \text{for any } \epsilon > 0$$

   Hint: Construct a random variable with certain distribution and then apply some inequalities.

   (One more problem on the 2nd page)

5. (25 points) Let X be a discrete random variable with pmf

$$f_\theta(x) = \frac{\gamma(x)\theta^x}{\phi(\theta)}, \qquad x = 0, 1, 2, ...$$

where $\theta > 0$ is a fixed constant, $\gamma(x) \geq 0$ is a function of x, and

$$\phi(\theta) = \sum_{x=0}^{\infty} \gamma(x)\theta^x$$

is assumed to be finite for any $\theta > 0$.

(a) Show that the family of pmf's, $\{f_\theta, \theta > 0\}$, is an exponential family.

(b) Obtain the moment generating function $M_X(t)$ in terms of a function of t and θ.

(c) Obtain $E(X)$ by differentiating the moment generating function.

(d) Show that $E(X)$ can also be obtained by differentiating $\phi(\theta)$ and interchanging the differentiation and summation (you need to justify why they can be interchanged).

(e) Suppose that $X_1, ..., X_n$ are independent and identically distributed with $f_\theta(x)$ and $S = \sum_{i=1}^n X_i$. Using the moment generating function, show that the pmf of S is

$$g_\theta(s) = \frac{\gamma_n(s)\theta^s}{[\phi(\theta)]^n}, \qquad s = 0, 1, 2, ...$$

where $\gamma_n(s) \geq 0$ is a function of s defined by

$$[\phi(\theta)]^n = \sum_{s=0}^{\infty} \gamma_n(s)\theta^s$$