STAT 609 Second Exam
2:30pm-3:45pm, Nov. 20, 2015

Please show all your work for full credits.

1. Let X and Y be random variables having the joint pdf

$$f(x,y) = \begin{cases} x+y & 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) (5 points) Find $\text{Cov}(X,Y)$.
   (b) (5 points) Find $E(X|Y=y)$ for $0<y<1$.
   (c) (5 points) Find $E(Ye^{X+X^{-1}}|X=x)$ for $0<x<1$.
   (d) (10 points) Find the mgf $M(t,s)=E(e^{tX+sY})$. (Hint: integration by parts)

2. Suppose that the 2-dimensional random vector (X,Y) is bivariate normal with pdf

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right\} \qquad (x,y)\in\mathcal{R}^2$$

   (a) (5 points) Find the conditional pdf of Y given X.
   (b) (5 points) Show that $X-Y$ and $X+Y$ are independent.
   (c) (10 points) Let $U=\max\{X,Y\}$ and $V=\min\{X,Y\}$. Find the joint pdf of (V,U). (Hint: Show $P(U\le u, V\le v) = P(X\le u, Y\le u) - P(v\le X\le u, v\le Y\le u)$.)
   (d) (10 points) Find the pdf of $R=U-V$.

3. Let Z be a random variable having pdf

$$f_Z(z) = \begin{cases} e^{-z} & 0<z \\ 0 & \text{otherwise} \end{cases}$$

   Let X and Y be random variables such that given $Z=z$, X and Y are conditionally independent with conditional pdf's

$$f_X(x|z) = ze^{-zx}, \qquad f_Y(y|z) = ze^{-zy}, \qquad x>0,\ y>0$$

   (a) (10 points) Find the joint pdf of (X,Y). (Hint: $\int_0^\infty t^{k-1}e^{-t}dt = \Gamma(k) = (k-1)!$)
   (b) (10 points) Calculate $E(Z|X=x,Y=y)$.

   (One more problem on the 2nd page)

4. Let $(X_1, ..., X_n)$ be a random sample from a population (the cdf of X_1) which has a pdf

$$f_\theta(x) = \frac{1-\theta^2}{2} \exp(\theta x - |x|), \qquad x \in \mathcal{R}$$

where θ is an unknown parameter satisfying $|\theta| < 1$.

(a) (10 points) Find the joint pdf $f(x, y)$ of $(X_{(1)}, X_{(n)})$, $x < y$, where $X_{(j)}$ is the jth order statistic. Express $f(x, y)$ explicitly in three regions: $\{0 < x < y\}$, $\{x < y < 0\}$, and $\{x < 0 < y\}$.

(b) (5 points) Using Theorem 5.2.11, show that $T = X_1 + \cdots + X_n$ has a pdf from an exponential family.

(c) (10 points) For $n = 2$, show (without using Theorem 5.2.11) that the pdf of $T = X_1 + X_2$ is from an exponential family, using the formula for the pdf of a sum of two random variables. (Hint: you may leave an integral in the formula of the pdf and still argue that the pdf is from an exponential family.)