STAT 609 Final Exam
2:30-4:30pm, Dec. 20, 2015

Please show all your work for full credits.

1. (8 points) On the desk of a professor there are two boxes, B_1 and B_2, each containing 10 computer disks. Box B_1 contains 6 V disks and 4 C disks and box B_2 contains 2 V disks and 8 C disks. A box is selected with probabilities $P(B_1) = 3/4$ and $P(B_2) = 1/4$. A disk is then selected at random from the chosen box.

   (a) (5 points) Find the probability that the selected disk is a V disk.

   (b) (5 points) Given that the selected disk is a V disk, find the probability that it was from box B_1.

2. Suppose that the joint distribution of (X, Y, Z) is multivariate normal with mean vector $(0, 0, 0)$ and covariance matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

   (a) (5 points) Calculate the covariance between $3X+2Y$ and $X-3Z$. Are they independent?

   (b) (5 points) Calculate the covariance between $X - 2Y + 2Z$ and $X + Y + Z$. Are they independent?

   (c) (5 points) Show that $\frac{(X+Y-Z)^2}{c}$ has the central chi-square distribution with degree of freedom 1, and find the value of c.

   (d) (5 points) Show that $\frac{(X+Y-Z)^2}{c} + \frac{(2X-Y+Z)^2}{d}$ has the central chi-square distribution with degrees of freedom 2, and find the value of d.

   (e) (5 points) Find the conditional distribution of (X, Y) given $Z = z$.

3. Let the joint pdf of (X, Y) be

$$f(x, y) = \begin{cases} x + y & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Let $U = X + Y$ and $V = X - Y$.

   (a) (5 points) Show that the joint pdf of (U, V) is

$$g(u, v) = \begin{cases} u/2 & 0 < u < 1,\ -u < v < u \\ u/2 & 1 < u < 2,\ u - 2 < v < 2 - u \\ 0 & \text{otherwise} \end{cases}$$

   (b) (5 points) Obtain the conditional pdf of V given $U = u$.

   (c) (5 points) For any given t, show that

$$E(e^{tV} | U = u) = \begin{cases} \frac{e^{tu} - e^{-tu}}{2tu} & 0 < u < 1 \\ \frac{e^{t(2-u)} - e^{t(u-2)}}{2t(2-u)} & 1 < u < 2 \end{cases}$$

(continued on the next page)

(d) (5 points) Use the result in the previous part to show that the mgf of V is

$$M_V(t) = E(e^{tV}) = \frac{e^t + e^{-t} - 2}{t^2}$$

4. Let X_n be a random variable having the chi-square distribution with degrees of freedom n, $n = 1, 2, ...$ In the following, the limiting process is with respect to $n \to \infty$.

   (a) (6 points) Show that

   $$\frac{X_n}{\sqrt{2n}} - \sqrt{\frac{n}{2}} \text{ converges in distribution to } N(0,1)$$

   (b) (6 points) For any positive integer k, show that

   $$\frac{X_n^k}{\sqrt{2}n^{k-1/2}} - \sqrt{\frac{n}{2}} \text{ converges in distribution to } N(0,k^2)$$

   (c) (6 points) Define

   $$Y_n = \begin{cases} 0 & X_n \leq 1 \\ X_n & X_n > 1 \end{cases}$$

   Show that $Y_n - X_n$ converges in probability to 0.

5. Let $X_1, ..., X_n$ be a random sample with a pdf

   $$f(x) = \begin{cases} \theta\varphi^\theta x^{-(\theta+1)} & x > \varphi \\ 0 & x \leq \varphi \end{cases}$$

   where $\theta > 0$ and $\varphi > 0$ are fixed parameters.

   (a) (5 points) Find the pdf of the jth order statistic, $j = 1, ..., n$.

   (b) (7 points) When φ is known and θ is unknown, show that the family of pdf's indexed by θ is an exponential family and find a sufficient and complete statistic T for θ.

   (c) (7 points) When θ is known and φ is unknown, show that the minimum order statistic $X_{(1)}$ is a complete and sufficient statistic for φ.

   (d) (8 points) Suppose that $\varphi = e^\theta$ is unknown. Show that the pair of statistics, T and $X_{(1)}$, is minimal sufficient for θ.