Lecture 7: Interchange of integration and limit

Differentiating under an integral sign

- To study the properties of a chf, we need some technical result.
- When can we switch the differentiation and integration?
- If the range of the integral is finite, this switch is usually valid.

Theorem 2.4.1 (Leibnitz's rule)

If $f(x,\theta)$, $a(\theta)$, and $b(\theta)$ are differentiable with respect to θ, then

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx = f(b(\theta),\theta)b'(\theta) - f(a(\theta),\theta)a'(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial f(x,\theta)}{\partial\theta}dx$$

In particular, if $a(\theta) = a$ and $b(\theta) = b$ are constants, then

$$\frac{d}{d\theta}\int_a^b f(x,\theta)dx = \int_a^b \frac{\partial f(x,\theta)}{\partial\theta}dx$$

- If the range of integration is infinite, problems can arise.
- Whether differentiation and integration can be switched is the same as whether limit and integration can be interchanged.

Interchange of integration and limit

Note that

$$\int_{-\infty}^{\infty} \frac{\partial f(x,\theta)}{\partial \theta} dx = \int_{-\infty}^{\infty} \lim_{\delta \to 0} \frac{f(x,\theta+\delta) - f(x,\theta)}{\delta} dx$$

Hence, the interchange of differentiation and integration means whether this is equal to

$$\frac{d}{d\theta} \int_{-\infty}^{\infty} f(x,\theta) dx = \lim_{\delta \to 0} \int_{-\infty}^{\infty} \frac{f(x,\theta+\delta) - f(x,\theta)}{\delta} dx$$

An example of invalid interchanging integration and limit

Consider

$$f(x,\delta) = \begin{cases} \delta^{-1} & 0 < x < \delta \\ 0 & \text{otherwise} \end{cases}$$

Then, $\lim_{\delta \to 0} f(x,\delta) = 0$ for any x and, hence,

$$0 = \int_{-\infty}^{\infty} \lim_{\delta \to 0} f(x,\delta) dx \neq \lim_{\delta \to 0} \int_{-\infty}^{\infty} f(x,\delta) dx = \lim_{\delta \to 0} \int_{0}^{\delta} \delta^{-1} dx = 1$$

We now give some sufficient conditions under which interchanging integration and limit (or differentiation) is valid.

The proof of the main result is technical and out of the scope of this course.

Theorem 2.4.2 (Lebesgue's dominated convergence theorem)

Suppose that the function $h(x,y)$ is continuous at y_0 for each x, and there exists a function $g(x)$ satisfying
(i) $|h(x,y)| \le g(x)$ for all x and y in a neighborhood of y_0;
(ii) $\int_{-\infty}^{\infty} g(x)dx < \infty$.
Then,

$$\lim_{y\to y_0} \int_{-\infty}^{\infty} h(x,y)dx = \int_{-\infty}^{\infty} \lim_{y\to y_0} h(x,y)dx$$

The key condition in this theorem is the existence of a dominating function $g(x)$ with a finite integral.
Applying Theorem 2.4.2 to the difference $[f(x,\theta+\delta)-f(x,\theta)]/\delta$, we obtain the next result.

Theorem 2.4.3.

Suppose that $f(x,\theta)$ is differentiable at $\theta = \theta_0$, i.e.,

$$\lim_{\delta\to 0} \frac{f(x,\theta_0+\delta)-f(x,\theta_0)}{\delta} = \left.\frac{\partial f(x,\theta)}{\partial\theta}\right|_{\theta=\theta_0}$$

exists for every x, and there exists a function $g(x,\theta_0)$ and a constant $\delta_0 > 0$ such that

(i) $\left|\frac{f(x,\theta_0+\delta)-f(x,\theta_0)}{\delta}\right| \le g(x,\theta_0)$ for all x and $|\delta| \le \delta_0$;

(ii) $\int_{-\infty}^{\infty} g(x,\theta_0)dx < \infty$.

Then

$$\frac{d}{d\theta}\int_{-\infty}^{\infty} f(x,\theta)dx\Big|_{\theta=\theta_0} = \int_{-\infty}^{\infty} \left.\frac{\partial f(x,\theta)}{\partial\theta}\right|_{\theta=\theta_0} dx$$

This result is for θ at a particular value, θ_0.

Typically $f(x,\theta)$ is differentiable at all θ, and

$$\frac{f(x,\theta_0+\delta)-f(x,\theta_0)}{\delta} = \left.\frac{\partial f(x,\theta)}{\partial\theta}\right|_{\theta=\theta_0+\delta^*(x)}$$

for some $\delta^*(x)$ with $|\delta^*(x)| \le \delta_0$, which leads to the next result.

Corollary 2.4.4.

Suppose that $f(x,\theta)$ is differentiable in θ and there exists a function $g(x,\theta)$ such that
(i) $\left|\frac{\partial f(x,\vartheta)}{\partial \vartheta}\right| \le g(x,\theta)$ for all x and ϑ such that $|\vartheta-\theta| \le \delta_0$;
(ii) $\int_{-\infty}^{\infty} g(x,\theta)dx < \infty$ for all θ.
Then

$$\frac{d}{d\theta}\int_{-\infty}^{\infty} f(x,\theta)dx = \int_{-\infty}^{\infty} \frac{\partial f(x,\theta)}{\partial \theta}dx$$

Example 2.4.5.

Let X have the exponential pdf

$$f(x) = \lambda^{-1}e^{-x/\lambda}, \quad x > 0$$

where $\lambda > 0$ is a constant.
We want to prove a recursion relation

$$E(X^{n+1}) = \lambda E(X^n) + \lambda \frac{d}{d\lambda}E(X^n), \qquad n = 1, 2, ...$$

Note that both $E(X^n)$ and $E(X^{n+1})$ are functions of λ.

If we could move the differentiation inside the integral, we would have

$$\begin{aligned}\frac{d}{d\lambda}E(X^n) &= \frac{d}{d\lambda}\int_0^\infty \frac{x^n}{\lambda}e^{-x/\lambda}dx = \int_0^\infty \frac{\partial}{\partial\lambda}\left(\frac{x^n}{\lambda}e^{-x/\lambda}\right)dx \\ &= \int_0^\infty \frac{x^n}{\lambda^2}\left(\frac{x}{\lambda}-1\right)e^{-x/\lambda}dx = \frac{1}{\lambda^2}E(X^{n+1}) - \frac{1}{\lambda}E(X^n)\end{aligned}$$

which is the result we want to show.

To justify the interchange of integration and differentiation, we take

$$g(x,\lambda) = \frac{x^n e^{-x/(\lambda+\delta_0)}}{(\lambda-\delta_0)^2}\left(\frac{x}{\lambda-\delta_0}+1\right)$$

Then

$$\left|\frac{\partial}{\partial\xi}\left(\frac{x^n}{\xi}e^{-x/\xi}\right)\right| \le \frac{x^n e^{-x/\xi}}{\xi^2}\left|\frac{x}{\xi}+1\right| \le g(x,\lambda), \quad |\xi-\lambda| \le \delta_0$$

and we can apply Corollary 2.4.4.

In the proof of Theorem 2.3.7 (differentiating mgf to obtain moments), we interchanged differentiation and integration without justification. We now provide a proof.

Completion of the proof of Theorem 2.3.7.

Assuming that $M_X(t)$ exists for $t \in [-h, h]$, $h > 0$, we want to show

$$\frac{d}{dt}M_X(t) = \frac{d}{dt}\int_{-\infty}^{\infty} e^{tx} f_X(x)dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}e^{tx} f_X(x)dx$$

It is enough to show (why?)

$$\frac{d}{dt}\int_{0}^{\infty} e^{tx} f_X(x)dx = \int_{0}^{\infty} \frac{\partial}{\partial t}e^{tx} f_X(x)dx$$

For $t \in (-h/2, h/2)$ and $x > 0$,

$$\left|\frac{\partial}{\partial t}e^{tx} f_X(x)\right| = \left|xe^{tx} f_X(x)\right| \le xe^{hx/2} f_X(x) = g(x)$$

For sufficiently large $x > 0$, $x \le e^{hx/2}$ and, hence,

$$\int_{0}^{\infty} g(x)dx \le \int_{0}^{\infty} e^{hx} f_X(x)dx = M_X(h) < \infty$$

An application of Corollary 2.4.4 proves the result.

We now complete the proof of Theorem C1 and establish another useful result for inverting a characteristic function.

Completion of the proof of Theorem C1.

We give a proof for the case where X has pdf $f_X(x)$.

If we can exchange the differentiation and integration, then

$$\frac{d^r\phi_X(t)}{dt^r} = \frac{d^r}{dt^r}\int_{-\infty}^{\infty} e^{\imath tx} f_X(x)dx = \int_{-\infty}^{\infty} \frac{d^r}{dt^r} e^{\imath tx} f_X(x)dx = \int_{-\infty}^{\infty} \imath^r x^r e^{\imath tx} f_X(x)dx$$

and the result follows by setting $t = 0$ at the beginning and end of the above expression.

The exchange is justified by the dominated convergence theorem, since

$$\left|\frac{d^r}{dt^r} e^{\imath tx} f_X(x)\right| = \left|\imath^r x^r e^{\imath tx}\right| f_X(x) \le |x|^r f_X(x)$$

and

$$E|X|^r = \int_{-\infty}^{\infty} |x|^r f_X(x)dx < \infty$$

Theorem C7.

Let $\phi(t)$ be a chf satisfying $\int_{-\infty}^{\infty} |\phi(t)|dt < \infty$.
Then the corresponding cdf F has derivative

$$F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\imath tx}\phi(t)dt$$

Proof.

Let $x-\delta$ and $x+\delta$ be continuity points of F.
By Theorem C2,

$$\begin{aligned} F(x+\delta)-F(x-\delta) &= \lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-it(x-\delta)}-e^{-it(x+\delta)}}{2\pi it}\phi(t)dt \\ &= \lim_{T\to\infty}\int_{-T}^{T}\frac{\sin(t\delta)e^{-itx}}{\pi t}\phi(t)dt \\ &= \int_{-\infty}^{\infty}\frac{\sin(t\delta)e^{-itx}}{\pi t}\phi(t)dt \end{aligned}$$

where the last equality follows from the fact that

$$\left|\frac{\sin(t\delta)e^{-itx}}{\pi t\delta}\phi(t)\right| \le |\phi(t)|$$

which is integrable.

Also, from the previous bound we can apply the dominated convergence theorem to interchange the integration and limit in the following operation to finish the proof:

$$\begin{aligned} F'(x) &= \lim_{\delta\to 0} \frac{F(x+\delta)-F(x-\delta)}{2\delta} \\ &= \lim_{\delta\to 0} \int_{-\infty}^{\infty} \frac{\sin(t\delta)e^{-itx}}{2\pi t\delta}\phi(t)dt \\ &= \int_{-\infty}^{\infty} \lim_{\delta\to 0} \frac{\sin(t\delta)e^{-itx}}{2\pi t\delta}\phi(t)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)dt \end{aligned}$$

Example

We now apply Theorem C7 to obtain the pdf corresponding to the chf $\phi(t) = e^{-|t|}$:

$$F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}e^{-|t|}dt = \frac{1}{\pi}\int_{0}^{\infty} \cos(tx)e^{-t}dt$$

Since $\int e^{at}\cos(bt)dt = e^{at}[a\cos(bt)+b\sin(bt)]/(a^2+b^2)$,

$$F'(x) = \frac{1}{\pi}\frac{e^{-t}[-\cos(xt)+x\sin(xt)]}{1+x^2}\Bigg|_{t=0}^{t=\infty} = \frac{1}{\pi(1+x^2)}$$

We now turn to the question of when we can interchange differentiation and infinite summation.

Theorem 2.4.8.

Suppose that the series $\sum_{x=0}^{\infty} h(\theta, x)$ converges for all θ in an interval $(a, b) \subset \mathscr{R}$ and

(i) $\frac{\partial}{\partial\theta} h(\theta, x)$ is continuous in θ for each x;

(ii) $\sum_{x=0}^{\infty} \frac{\partial}{\partial\theta} h(\theta, x)$ converges uniformly on every closed bounded subinterval of (a, b).

Then

$$\frac{d}{d\theta} \sum_{x=0}^{\infty} h(\theta, x) = \sum_{x=0}^{\infty} \frac{\partial}{\partial\theta} h(\theta, x)$$

This theorem is from calculus (or mathematical analysis) and is not easy to apply because we need to show the uniform convergence of an infinite series.

It is more convenient to apply the following dominated convergence theorem for infinite sums.

Theorem 2.4.2A (dominated convergence theorem)

Suppose that the function $h(x,y)$ is continuous at y_0 for each x, and there exists a function $g(x)$ satisfying
(i) $|h(x,y)| \le g(x)$ for all x and y in a neighborhood of y_0;
(ii) $\sum_x g(x) < \infty$.
Then,

$$\lim_{y \to y_0} \sum_x h(x,y) = \sum_x \lim_{y \to y_0} h(x,y)$$

If $h(x,\theta)$ is differentiable in θ and there is a function $g(x,\theta)$ such that
(i) $\left|\frac{\partial h(x,\vartheta)}{\partial \vartheta}\right| \le g(x,\theta)$ for all x and ϑ such that $|\vartheta - \theta| \le \delta_0$;
(ii) $\sum_x g(x,\theta) < \infty$ for all θ.
Then

$$\frac{d}{d\theta} \sum_x h(x,\theta) = \sum_x \frac{\partial h(x,\theta)}{\partial \theta}$$

Example 2.4.7.

Let X be a discrete random variable having pmf

$$f_X(x) = P(X = x) = \theta(1-\theta)^x, \quad x = 0,1,2,...$$

where $\theta \in (0,1)$ is a fixed constant.

We want to derive $E(X)$.

Note that

$$\sum_{x=0}^{\infty} \theta(1-\theta)^x = 1$$

If interchange of summation and differentiation is justified,

$$\begin{aligned} 0 &= \frac{d}{d\theta} \sum_{x=0}^{\infty} \theta(1-\theta)^x = \sum_{x=0}^{\infty} \frac{\partial}{\partial \theta} \theta(1-\theta)^x \\ &= \sum_{x=0}^{\infty} [(1-\theta)^x - \theta x (1-\theta)^{x-1}] \\ &= \frac{1}{\theta} \sum_{x=0}^{\infty} \theta(1-\theta)^x - \frac{1}{1-\theta} \sum_{x=0}^{\infty} x\theta(1-\theta)^x \\ &= \frac{1}{\theta} - \frac{1}{1-\theta} E(X) \end{aligned}$$

Thus, $E(X) = (1-\theta)/\theta$.

To justify the interchange, let $h(x,\theta) = \theta(1-\theta)^x$ and then

$$\frac{\partial}{\partial \theta} h(\theta, x) = (1-\theta)^x - \theta x (1-\theta)^{x-1}$$

Consider $\theta \in [c,d] \subset (0,1)$.

$$|(1-\theta)^x - \theta x(1-\theta)^{x-1}| \leq (1-c)^x + dx(1-c)^{x-1} = g(x)$$

Since

$$\sum_{x=0}^{\infty} [(1-c)^x + dx(1-c)^{x-1}] < \infty$$

the exchange of sum and differentiation is justified.

Monotone convergence theorem

In some cases, the following result is more convenient to apply.

Suppose that functions $g_n(x)$, $n = 1, 2, ...$, $x \in \mathscr{R}$, satisfying that $0 \leq g_n(x) \leq g_{n+1}(x)$ for all n and x and $\lim_{n\to\infty} g_n(x) = g(x)$ for all x. Then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)dx = \int_{-\infty}^{\infty} \lim_{n\to\infty} g_n(x)dx = \int_{-\infty}^{\infty} g(x)dx$$

which holds even if $g(x)$ is not integrable (i.e., both sides are infinity).
The same result holds when $\int$ is replaced by $\sum$ for discrete x.

Proof.

We prove the continuous case, since the discrete case is similar.

If $\int_{-\infty}^{\infty} g(x)dx < \infty$, then this result is a direct consequence of the dominated convergence theorem.

Suppose now that $\int_{-\infty}^{\infty} g(x)dx = \infty$.
Then

$$\lim_{M\to\infty} \int_{-\infty}^{\infty} g(x) I_{\{|x|\le M, g(x)\le M\}} dx = \infty$$

where I_A is the indicator function of A.

For each $M > 0$,

$$\int_{-\infty}^{\infty} g(x) I_{\{|x|\le M, g(x)\le M\}} dx \le 2M^2 < \infty$$

and $g_n(x) I_{\{|x|\le M, g(x)\le M\}}$ is monotone and converges to $g(x) I_{\{|x|\le M, g(x)\le M\}}$.
Then, for every M.

$$\begin{aligned}\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)dx &\ge \lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x) I_{\{|x|\le M, g(x)\le M\}} dx \\ &= \int_{-\infty}^{\infty} g(x) I_{\{|x|\le M, g(x)\le M\}} dx\end{aligned}$$

Since the right side goes to ∞,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)dx = \infty$$