Lecture 9: Characteristic functions

The mgf's are useful, but they sometimes are not finite.

Another function that can characterise a distribution is the so-called characteristic function, which is always well defined but it is a complex function.

Definition (characteristic functions).

The characteristic function (chf) of the distribution of a random variable X is defined as

$$\phi_X(t) = E(e^{itX}) = E[\cos(tX)] + \imath E[\sin(tX)], \qquad t \in \mathscr{R}$$

where $\imath = \sqrt{-1}$ and $e^{is} = \cos(s) + \imath \sin(s)$.

- If the mgf $M_X(t)$ of X is finite in a neighborhood of 0, then the chf of X is $\phi_X(t) = M_X(it)$, $t \in \mathscr{R}$.
- If $Y = aX + b$ for constants a and b, then the chf of Y in terms of the chf $\phi_X(t)$ of X is $\phi_Y(t) = e^{ibt}\phi_X(at)$, $t \in \mathscr{R}$.
- A chf is uniformly continuous in $\mathscr{R}$.

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- A chf is uniformly continuous in $\mathscr{R}$.

Theorem C1.

If a random variable X has finite $E|X|^r$ for a positive integer r, then

$$\left.\frac{d^r\phi_X(t)}{dt^r}\right|_{t=0} = i^r E(X^r)$$

Proof.

We give a proof for the case where X has pdf $f_X(x)$.

If we can exchange the differentiation and integration, then

$$\frac{d^r\phi_X(t)}{dt^r} = \frac{d^r}{dt^r}\int_{-\infty}^{\infty} e^{itx} f_X(x)dx = \int_{-\infty}^{\infty}\frac{d^r}{dt^r}e^{itx} f_X(x)dx = \int_{-\infty}^{\infty} i^r x^r e^{itx} f_X(x)dx$$

and the result follows by setting $t = 0$ at the beginning and end of the above expression.

The exchange is justified by the dominated convergence theorem, since

$$\left|\frac{d^r}{dt^r}e^{itx} f_X(x)\right| = \left|i^r x^r e^{itx}\right| f_X(x) \le |x|^r f_X(x)$$

and

$$E|X|^r = \int_{-\infty}^{\infty} |x|^r f_X(x)dx < \infty$$

Thus, like the mgf, the chf can be used to calculate the moments, but we have to first know the existence of moments.

Example.

Let X be a random variable with pdf

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \qquad x \in \mathscr{R}$$

From the Fourier transformation,

$$\int_0^\infty \frac{\cos(tx)}{1+x^2} dx = \frac{\pi}{2} e^{-t}, \qquad t > 0$$

we obtain that, when $t > 0$,

$$\begin{aligned} \phi_X(t) &= E[\cos(tX)] + iE[\sin(tX)] \\ &= \int_{-\infty}^\infty \frac{\cos(tx)}{\pi(1+x^2)} dx + i\int_{-\infty}^\infty \frac{\sin(tx)}{\pi(1+x^2)} dx \\ &= \frac{2}{\pi} \int_{-\infty}^\infty \frac{\cos(tx)}{1+x^2} dx = e^{-t} \end{aligned}$$

since $\sin(s)$ is an odd function and $\cos(s)$ is an even function.

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since $\sin(s)$ is an odd function and $\cos(s)$ is an even function.

For $t < 0$,

$$\phi_X(t) = E[\cos(tX)] + \imath E[\sin(tX)] = E[\cos(-tX)] = e^t = e^{-|t|}$$

When $t = 0$, $\phi_X(0) = 1$ for any chf, and hence we obtain that

$$\phi_X(t) = e^{-|t|} \qquad t \in \mathscr{R}$$

Note that this chf is not differentiable at $t = 0$ and previously we showed that the expectation of X does not exist.

Uniqueness and inversion

Similar to the mgf, the chf characterizes the distribution in the sense that there is a one-to-one correspondence between chf and cdf.

We first establish an inversion formula.

Theorem C2

Let X be a random variable with cdf F and chf ϕ.

For any real numbers $y < u$,

$$P(y < X < u) + \frac{P(X = y) + P(X = u)}{2} = \lim_{T \to \infty} \int_{-T}^{T} \frac{e^{-\imath ty} - e^{-\imath tu}}{2\pi \imath t} \phi(t) dt$$

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Proof.

By definition,

$$\begin{aligned}\int_{-T}^{T} \frac{e^{-ity}-e^{-itu}}{2\pi it}\phi(t)dt &= \int_{-T}^{T} \frac{e^{-ity}-e^{-itu}}{2\pi it}E(e^{itX})dt \\ &= E\int_{-T}^{T} \frac{e^{-ity}-e^{-itu}}{2\pi it}e^{itX}dt \\ &= E[g_T(u,y,X)]\end{aligned}$$

where the switch of E and $\int_{-T}^{T}$ is justified by the fact that

$$\left|\frac{e^{-ity}-e^{-itu}}{it}e^{itX}\right| = \left|\int_{y}^{u} e^{-its}ds\right| \leq (u-y)$$

and

$$\begin{aligned}g_T(u,y,x) &= \int_{-T}^{T} \frac{e^{it(x-y)}-e^{it(x-u)}}{2\pi it}dt \\ &= \frac{1}{\pi}\int_{0}^{T} \frac{\sin(t(x-y))-\sin(t(x-u))}{t}dt\end{aligned}$$

because $\frac{\sin(t(x-y))-\sin(t(x-u))}{t}$ is even and $\frac{\cos(t(x-y))-\cos(t(x-u))}{t}$ is odd.

By Fourier transformation,

$$\int_0^\infty \frac{\sin(st)}{t}dt = \frac{\pi}{2}\mathrm{sign}(s)$$

Hence

$$\lim_{T\to\infty} g_T(u,y,x) = \begin{cases} -\frac{1}{2}-(-\frac{1}{2}) = 0 & x < y \\ 0-(-\frac{1}{2}) = \frac{1}{2} & x = y \\ \frac{1}{2}-(-\frac{1}{2}) = 1 & y < x < u \\ \frac{1}{2}-0 = \frac{1}{2} & x = u \\ \frac{1}{2}-\frac{1}{2} = 0 & x > u \end{cases}$$

Also,

$$0 \le \mathrm{sign}(s)\int_0^T \frac{\sin(st)}{t}dt \le \int_0^\pi \frac{\sin t}{t}dt \qquad T > 0$$

which implies that

$$|g_T(u,y,x)| \le \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}dt \qquad T > 0$$

By the dominated convergence theorem,

$$
\begin{aligned}
\lim_{T\to\infty} E\left[g_T(u,y,X)\right] &= E\left[\lim_{T\to\infty} g_T(u,y,X)\right] \\
&= E\left[I(y<X<u)+\frac{1}{2}I(X=y)+\frac{1}{2}I(X=u)\right] \\
&= P(y<X<u)+\frac{P(X=y)+P(X=u)}{2}
\end{aligned}
$$

This proves the theorem.

Theorem C3.

Let F be a cdf with chf ϕ.

(i) If F is continuous at y and u with $y<u$, then

$$F(u)-F(y)=\lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-ity}-e^{-itu}}{2\pi it}\phi(t)dt$$

(ii) If F is continuous at x, then

$$F(u)=\lim_{y\to-\infty}\lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-ity}-e^{-itu}}{2\pi it}\phi(t)dt$$

(iii) There is a one-to-one correspondence between cdf and chf.

$$\begin{aligned}
\lim_{T\to\infty} E[g_T(u,y,X)] &= E\left[\lim_{T\to\infty} g_T(u,y,X)\right] \\
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Proof.

(i) is obvious.
(ii)-(iii) follow from the fact that any cdf can only have countably many discontinuous points and, hence, it is determined by the F values at continuity points of F.

If a cdf is continuous, then Theorem 3C(ii) gives an inversion formula.
For a discrete cdf, we have the following result whose proof is similar to that of Theorem 2C and is omitted.

Theorem C4.

Let F be a discrete cdf with chf ϕ.

(i) If F has a jump at x, then

$$F(x) - F(x-) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-itx} \phi(t) dt$$

(ii) If D is the set of all discontinuous points of F, then

$$\sum_{x\in D} [F(x) - F(x-)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\phi(t)|^2 dt$$

Proof.

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Theorem 4C(ii) gives a sufficient and necessary condition for a cdf to be continuous.

We know that $e^{-|t|}$ is a chf, and

$$\int_{-T}^{T}(e^{-|t|})^2 dt \le \int_{-\infty}^{\infty} e^{-2|t|}dt = 2\int_0^{\infty} e^{-2t}dt = 1 \qquad T > 0$$

Hence, the corresponding cdf must be continuous.

In fact, in this case we know that the cdf corresponding to $e^{-|t|}$ has a pdf according to the following result, which gives a sufficient condition for a cdf having a pdf.

Theorem C5.

Let $\phi(t)$ be a chf satisfying $\int_{-\infty}^{\infty}|\phi(t)|dt < \infty$. Then the corresponding cdf F has derivative

$$F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)dt$$

Proof.

Let $x - \delta$ and $x + \delta$ be continuity points of F.

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Proof.

Let $x - \delta$ and $x + \delta$ be continuity points of F.

By Theorem C2,

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\begin{aligned}
F(x+\delta)-F(x-\delta) &= \lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-it(x-\delta)}-e^{-it(x+\delta)}}{2\pi it}\phi(t)dt \\
&= \lim_{T\to\infty}\int_{-T}^{T}\frac{\sin(t\delta)e^{-itx}}{\pi t}\phi(t)dt \\
&= \int_{-\infty}^{\infty}\frac{\sin(t\delta)e^{-itx}}{\pi t}\phi(t)dt
\end{aligned}
$$

where the last equality follows from the fact that

$$
\left|\frac{\sin(t\delta)e^{-itx}}{\pi t\delta}\phi(t)\right| \leq |\phi(t)|
$$

which is integrable.

Also, from the previous bound we can apply the dominated convergence theorem to obtain that

$$
\begin{aligned}
F'(x) &= \lim_{\delta\to 0}\frac{F(x+\delta)-F(x-\delta)}{2\delta} = \lim_{\delta\to 0}\int_{-\infty}^{\infty}\frac{\sin(t\delta)e^{-itx}}{2\pi t\delta}\phi(t)dt \\
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Theorem C6.

A random variable X is symmetric about 0 iff its chf ϕ_X is real-valued.

Proof.

- If X and $-X$ have the same distribution, then $\phi_X(t) = \phi_{-X}(t)$.
  But $\phi_{-X}(t) = \phi_X(-t)$.
  Then $\phi_X(t) = \phi_X(-t)$.
  Note that $\sin(-tX) = -\sin(tX)$ and $\cos(tX) = \cos(-tX)$.
  Hence $E[\sin(tX)] = 0$ and, thus, ϕ_X is real-valued.
- If ϕ_X is real-valued, then $\phi_X(t) = E[\cos(tX)]$ and
  $\phi_{-X}(t) = \phi_X(-t) = \phi_X(t)$.
  By Theorem C3, X and $-X$ must have the same distribution.

Theorem C7.

Suppose that $X_1, X_2, \dots$ is a sequence of random variables with chf's $\phi_{X_n}(t)$, and $\lim_{n\to\infty} \phi_{X_n}(t) = \phi(t)$, $t \in \mathscr{R}$, where $\phi(t)$ is continuous in t. Then ϕ must be the chf of a random variable X and for all x at which $F_X(x)$ is continuous,

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$

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