Chapter 4: Multiple Random Variables
Lecture 10: Joint and conditional distributions

So far we focus on the distribution of a single random variable.

In applications we need to consider a set of random variables jointly.

In some cases we study relationships among random variables.

Definition 4.1.1.

For an integer n, $X = (X_1, ..., X_n)$ is called an n-dimensional random vector iff each X_i is a random variable.

Joint cdf's

The joint cdf of an n-dimensional random vector X is a function F_X in $\mathscr{R}^n$ such that

$$\begin{aligned} F_X(x_1,...,x_n) &= P\left(\bigcap_{i=1}^n \{X_i \le x_i\}\right) & x_i \in \mathscr{R} \\ &= P(X_1 \le x_1, ..., X_n \le x_n) & i = 1, ..., n \end{aligned}$$

Properties of cdf's

- F_X is nondecreasing and right-continuous in any of its n arguments.
- For any $i = 1, ..., n$ and fixed $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n$,

$$\lim_{x_i \to -\infty} F(x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n) = 0$$

$$\lim_{x_i \to \infty,\ i=1,...,n} F(x_1, ..., x_n) = 1$$

Marginal cdf's

For a random vector $X = (X_1, ..., X_n)$ and any i, the cdf of X_i is called the marginal cdf of X_i and is equal to

$$F_{X_i}(x_i) = \lim_{x_j \to \infty,\ j=1,...,i-1,i+1,...,n} F_X(x_1, ..., x_n)$$

Knowing the joint cdf F_X we can obtain n marginal cdf's, but in general, knowing $F_{X_1}, ..., F_{X_n}$ is not enough to determine the joint cdf F_X.

Similar to the univariate case, we mainly consider two types of random vectors, discrete random vectors and continuous random vectors.

Definition 4.1.3 (Discrete joint pmf)

A random vector $X = (X_1,...,X_n)$ is discrete iff each X_i is discrete.
The joint pmf of X is

$$f_X(x_1,...,x_n) = P(X_1 = x_1,...,X_n = x_n),$$

which is positive only for countably many $(x_1,...,x_n) \in \mathscr{R}^n$.

- For any event $A \subset \mathscr{R}^n$,

$$P(X \in A) = \sum_{(x_1,...,x_n)\in A} f_X(x_1,...,x_n)$$

- For any i, the marginal pmf of X_i is

$$f_{X_i}(x_i) = \sum_{x_1,...,x_{i-1},x_{i+1},...,x_n} f_X(x_1,...,x_n)$$

- For any function $g(x_1,...,x_n)$, the expected value of $g(X_1,...,X_n)$ is

$$E[g(X_1,...,X_n)] = \sum_{x_1,...,x_n} g(x_1,...,x_n) f_X(x_1,...,x_n)$$

- If $n = 2$ and each X_i takes finitely many values, then the joint and marginal pmf's can be listed in a 2×2 table.

Values and joint and marginal pmf's of a 2-dimensional random vector

value	x_{21}	x_{22}	$\cdots$	x_{2c}	marginal
x_{11}	p_{11}	p_{12}	$\cdots$	p_{1c}	$p_{1\cdot}$
x_{12}	p_{21}	p_{22}	$\cdots$	p_{2c}	$p_{2\cdot}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_{1r}	p_{r1}	p_{r2}	$\cdots$	p_{rc}	$p_{r\cdot}$
marginal	$p_{\cdot 1}$	$p_{\cdot 2}$	$\cdots$	$p_{\cdot c}$	1

$$p_{ij} = P(X_1 = i, X_2 = j),\ p_{i\cdot} = P(X_1 = i),\ p_{\cdot j}(X_2 = j).$$

Example: Multinomial distribution

An experiment has r possible outcomes $A_1, ..., A_r$ with $P(A_i) = p_i$,
$i = 1, ..., r$, $p_1 + \cdots + p_r = 1$.
We independently repeat the experiment n times.
If X_i is the number of times A_i is the result in n experiments, $i = 1, ..., r$, then $X = (X_1, ..., X_r)$ has joint pmf

$$P(X_1 = x_1, ..., X_r = x_r) = \frac{n!}{x_1! \cdots x_r!} p_1^{x_1} \cdots p_r^{x_r} \qquad 0 \le x_j,\ \sum_{i=1}^{r} x_i = n$$

Multivariate hypergeometric distribution

A bag contains N balls with r different colors.
$N_i =$ the number of balls for color i, $N_1 + \cdots + N_r = N$.
We randomly select n balls from the bag.
If X_i is the number of selected balls having color i, $i = 1, ..., r$, then $X = (X_1, ..., X_r)$ has joint pmf

$$P(X_1 = x_1, ..., X_r = x_r) = \frac{\binom{N_1}{x_1} \cdots \binom{N_r}{x_r}}{\binom{N}{n}} \qquad 0 \le x_j, \ \sum_{i=1}^{r} x_i = n$$

What are the marginal pmf's for the multinomial and multivariate hypergeometric distributions?

Definition 4.1.10 (continuous pdf)

A random vector $X = (X_1, ..., X_n)$ has a continuous joint pdf if there exists a nonnegative function f_X on $\mathscr{R}^n$ such that

$$P(X \in A) = \int \cdots \int_{(x_1, ..., x_n) \in A} f_X(x_1, ..., x_n) dx_1 \cdots dx_n$$

- If $X = (X_1, ..., X_n)$ has joint pdf f_X, a short notation is

$$P(X \in A) = \int_A f_X(x)dx \qquad x = (x_1, ..., x_n)$$

- If the joint pdf f_X exists, then

$$F_X(x_1, ..., x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_X(t_1, ..., t_n)dt_1 \cdots dt_n$$

- If the joint cdf F_X is differentiable, then the joint f_X exists and

$$f_X(x_1, ..., x_n) = \frac{\partial^n F_X(x_1, ..., x_n)}{\partial x_1 \cdots \partial x_n}, \qquad (x_1, ..., x_n) \in \mathscr{R}^n$$

- If the joint pdf f_X exists, then the ith margianl pdf f_{X_i} exists and

$$f_{X_i}(x_i) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_X(x_1, ..., x_n)dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_n$$

- If the joint pdf f_X exists, then for any function $g(x_1, ..., x_n)$, the expected value of $g(X_1, ..., X_n)$ is

$$E[g(X_1, ..., X_n)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, ..., x_n)f_X(x_1, ..., x_n)dx_1 \cdots dx_n$$

Example

Suppose that a 2-dimensional random vector (X, Y) has pdf

$$f(x,y) = \begin{cases} Ce^{-(2x+3y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- What should C be? Since

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= \int_0^{\infty}\int_0^{\infty} Ce^{-(2x+3y)}dxdy \\ &= C\int_0^{\infty} e^{-2x}dx \int_0^{\infty} e^{-3y}dy \\ &= C\left(\frac{e^{-2x}}{2}\bigg|_{\infty}^{0} \frac{e^{-3y}}{3}\bigg|_{\infty}^{0}\right) = \frac{C}{6} \end{aligned}$$

  must be 1, we obtain that $C = 6$.

- The joint cdf of (X, Y) is

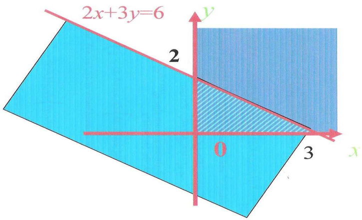
$$F(x,y) = \begin{cases} (1-e^{-2x})(1-e^{-3y}) & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

This is because, for $x > 0$ and $y > 0$,

$$F(x,y) = \int_0^\infty \int_0^\infty 6e^{-(2x+3y)} dxdy = 6\int_0^\infty e^{-2x} dx \int_0^\infty 6e^{-3y} dy$$
$$= (1-e^{-2x})(1-e^{-3y})$$

- Calculate $P(2X + 3Y \leq 6)$.
  It is not convenient to use the joint cdf.
  Using the joint pdf, we obtain

$$P(2X + 3Y \leq 6) = \int_{2x+3y\leq 6} f(x,y)dxdy$$



$$
\begin{aligned}
&= 6\int_0^3 \left[\int_0^{(6-2x)/3} e^{-(2x+3y)}dy\right]dx = 6\int_0^3 e^{-2x}\left[-\frac{e^{-3y}}{3}\bigg|_0^{(6-2x)/3}\right]dx \\
&= 6\int_0^3 e^{-2x}\left[\frac{1}{3}-\frac{e^{-(6-2x)}}{3}\right]dx = 2\int_0^3 (e^{-2x}-e^{-6})dx \\
&= -e^{-2x}\Big|_0^3 - 2e^{-6}\times 3 = 1-e^{-6}-6e^{-6} \\
&= 1-7e^{-6}
\end{aligned}
$$

Example 4.1.12.

Suppose that a 2-dimensional random vector (X, Y) has pdf

$$
f(x,y) = \begin{cases} e^{-y} & 0 < x < y \\ 0 & \text{otherwise} \end{cases}
$$

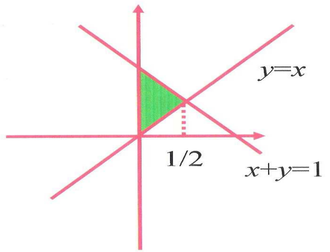
We want to calculate $P(X + Y \geq 1)$.

Note that

$$
\begin{aligned}
P(X+Y<1) &= \int_{x+y<1} f(x,y)dxdy = \int_0^{1/2}\left(\int_x^{1-x} e^{-y}dy\right)dx \\
&= \int_0^{1/2}(e^{-x}-e^{-(1-x)})dx = 1-e^{-1/2}-e^{-1/2}+e^{-1} \\
&= 1+e^{-1}-2e^{-1/2}
\end{aligned}
$$

Hence,

$$P(X+Y\geq 1) = 1-P(X+Y<1) = 2e^{-1/2}-e^{-1}$$



Example

Let

$$f(x,y) = \begin{cases} \frac{x^{\alpha-1}(y-x)^{\beta-1}e^{-y}}{\Gamma(\alpha)\Gamma(\beta)} & 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$ are constants.

We want to show this is a pdf and find its two marginal pdf's.

For $x > 0$, consider

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy = \int_x^{\infty} \frac{x^{\alpha-1}(y-x)^{\beta-1}e^{-y}}{\Gamma(\alpha)\Gamma(\beta)} dy \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\infty} u^{\beta-1}e^{-(u+x)}du \qquad y-x=u \\ &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x} \end{aligned}$$

This is the pdf of $Gamma(\alpha,1)$, which also shows that $f(x,y)$ is a pdf.

The other marginal pdf is, for $y > 0$,

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx = \int_0^y \frac{x^{\alpha-1}(y-x)^{\beta-1}e^{-y}}{\Gamma(\alpha)\Gamma(\beta)}dx \\ &= \frac{e^{-y}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 (ty)^{\alpha-1}(y-ty)^{\beta-1}ydt \qquad x = ty \\ &= \frac{y^{\alpha+\beta-1}e^{-y}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{y^{\alpha+\beta-1}e^{-y}}{\Gamma(\alpha+\beta)} \end{aligned}$$

which is the pdf of $Gamma(\alpha+\beta,1)$.

Definition 4.2.1 (conditional pmf)

Let $(X_1,...,X_n)$ be a discrete random vector with joint pmf $f(x)$ and k be an integer satisfying $1 \le k \le n-1$. The conditional pmf of $(X_{k+1},...,X_n)$ given that $(X_1,...,X_k) = (x_1,...,x_k)$ with $P(X_1 = x_1,...,X_k = x_k) > 0$ is

$$\begin{aligned} f(x_{k+1},...,x_n|x_1,...,x_k) &= P(X_{k+1} = x_{k+1},...,X_n = x_n|X_1 = x_1,...,X_k = x_k) \\ &= \frac{f(x_1,...,x_n)}{\sum_{(y_{k+1},...,y_n)\in\mathscr{N}_k} f(x_1,...,x_k,y_{k+1},...,y_n)} \end{aligned}$$

where $\mathscr{N}_k = \{(y_{k+1},...,y_n) : P(X_{k+1} = y_{k+1},...X_n = y_n) > 0\}$.

- It can be easily verified that $f(x_{k+1},...,x_n|x_1,...,x_k)$ is a pmf for any $(x_1,...,x_k)$ with $P(X_1 = x_1,...,X_k = x_k) > 0$.
- The conditional pmf $f(x_{k+1},...,x_n|x_1,...,x_k)$ vary with $x_1,...,x_k$.
- For any event $A \subset \mathscr{R}^{n-k}$,

$$
\begin{aligned}
& P((X_{k+1},...,X_n) \in A|X_1 = x_1,...,X_k = x_k) \\
= & \sum_{(x_{k+1},...,x_n)\in A} f(x_{k+1},...,x_n|x_1,...,x_k)
\end{aligned}
$$

Definition 4.2.3 (conditional pdf)

Let $(X_1,...,X_n)$ be a random vector with joint pdf $f(x)$ and k be an integer satisfying $1 \le k \le n-1$. The conditional pdf of $(X_{k+1},...,X_n)$ given that $(X_1,...,X_k) = (x_1,...,x_k)$ is

$$
f(x_{k+1},...,x_n|x_1,...,x_k) = \frac{f(x_1,...,x_n)}{\int_{\mathscr{R}^{n-k}} f(x_1,...,x_k,y_{k+1},...,y_n)dy_{k+1}\cdots dy_n}
$$

assuming that the denominator is not 0.

It can be easily verified that $f(x_{k+1},...,x_n|x_1,...,x_k)$ is a pdf and for any event $A \subset \mathscr{R}^{n-k}$,

$$P((X_{k+1},...,X_n) \in A|X_1 = x_1,...,X_k = x_k)$$
$$= \int_A f(x_{k+1},...,x_n|x_1,...,x_k)dx_{k+1}\cdots dx_n$$

In general, for random vectors X and Y (discrete or conditions), we use the notation $Y|X = x$ or $Y|X$ to denote the conditional distribution Y given $X = x$ or given X.

Example 4.2.4.

Suppose that a 2-dimensional random vector (X,Y) has pdf

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$

The marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_x^{\infty} e^{-y}dy = e^{-x}$$

if $x > 0$; $f_X(x) = 0$ if $x \le 0$; i.e., $X \sim \mathit{exponential}(0,1)$.

For each $x > 0$,

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{e^{-y}}{e^{-x}} = e^{-(y-x)} & y > x \\ 0 & y \leq x \end{cases}$$

i.e., $f(y|x)$ is the pdf of *exponential*$(x,1)$ for $x > 0$.
Thus, $Y|X = x \sim$ *exponential*$(x,1)$ or $Y|X \sim$ *exponential*$(X,1)$.

Conditional expectations

Let $(X_1,...,X_n)$ be a random vector with joint pmf or pdf $f(x)$, k be an integer satisfying $1 \leq k \leq n-1$, and g be a function on $\mathscr{R}^{n-k}$. The conditional expectation of $g(X_{k+1},...,X_n)$ given $(X_1,...,X_k) = (x_1,...,x_k)$ is

$$E[g(X_{k+1},...,X_n)|X_1 = x_1,...,X_k = x_k]$$
$$= \sum_{x_{k+1},...,x_n} g(x_{k+1},...,x_n) f(x_{k+1},...,x_n|x_1,...,x_k)$$

when f is a pmf and

$$= \int_{\mathscr{R}^{n-k}} g(x_{k+1},...,x_n) f(x_{k+1},...,x_n|x_1,...,x_k) dx_{k+1} \cdots dx_n$$

when f is a pdf.

- The condition expectation $E[g(X_{k+1},...,X_n)|X_1 = x_1,...,X_k = x_k]$ is a function of $x_1,...,x_k$.
- It is an expectation of the conditional distribution.
- Let $h(x_1,...,x_k) = E[g(X_{k+1},...,X_n)|X_1 = x_1,...,X_k = x_k]$. Then $h(X_1,...,X_k)$ is a random variable and is denoted by $E[g(X_{k+1},...,X_n)|X_1,...,X_k]$.

Example 4.2.4.

Since $Y|X \sim exponential(X,1)$, the conditional expectation of Y given X is $E(Y|X) = 1+X$, and the conditional expectation of $[Y - E(Y|X)]^2$ given X is

$$\begin{aligned} E\{[Y-E(Y|X)]^2|X\} &= \int_X^\infty [y-(1+X)]^2 e^{-y} dy \\ &= \int_X^\infty y^2 e^{-y} dy - (1+X)^2 \\ &= 1+(1+X)^2-(1+X)^2 = 1 \end{aligned}$$

Note that the function g may depend on X (treated as a constant).

Properties of conditional expectations

Conditional expectations have the following useful properties.
Let X, Y, and Z be random variables.

1. If $P(Y = c) = 1$ for a constant c, then $E(Y|X) = c$.
2. If $Y \leq Z$, then $E(Y|X) \leq E(Z|X)$.
3. For constants a and b, $E(aY + bZ|X) = aE(Y|X) + bE(Z|X)$.
4. $E[E(Y|X)] = E(Y)$ (Theorem 4.4.3). This can be interpreted as: the average of averages is the overall average.
5. $\mathrm{Var}(Y) = E[\mathrm{Var}(Y|X)] + \mathrm{Var}(E(Y|X))$ (Theorem 4.4.7), where $\mathrm{Var}(Y|X)$ is the variance of the conditional distribution $Y|X$.
6. For any function $g(X)$, $E[Yg(X)|X] = g(X)E(Y|X)$.

Except for property 2, all properties can be extended to random vectors X, Y, and Z with appropriate modifications on vector multiplications.

Proof of Theorem 4.4.3.

Consider the continuous case where (X, Y) has pdf $f(x, y)$.

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_{-\infty}^{\infty} y \left[\int_{-\infty}^{\infty} f(x, y) dx \right] dy$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} yf(x,y)dy\right]dx = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} y\frac{f(x,y)}{f_X(x)}dy\right]f_X(x)dx \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} yf(y|x)dy\right]f_X(x)dx = \int_{-\infty}^{\infty} E(Y|X=x)f_X(x)dx \\
&= E[E(Y|X)]
\end{aligned}
$$

Proof of Theorem 4.4.7.

Using properties 1, 3, 4, and 6, we obtain

$$
\begin{aligned}
\mathrm{Var}(Y) &= E[Y-E(Y)]^2 = E\left(E\{[Y-E(Y)]^2|X\}\right) \\
&= E\left(E\{[Y-E(Y|X)+E(Y|X)-E(Y)]^2|X\}\right) \\
&= E\left(E\{[Y-E(Y|X)]^2|X\}\right)+E\left(E\{[E(Y|X)-E(Y)]^2|X\}\right) \\
&\quad +2E\left(E\{[Y-E(Y|X)][E(Y|X)-E(Y)]|X\}\right) \\
&= E\left(\mathrm{Var}(Y|X)\right)+E\left([E(Y|X)-E(Y)]^2\right) \\
&\quad +2E\left([E(Y|X)-E(Y)]E\{[Y-E(Y|X)]|X\}\right) \\
&= E[\mathrm{Var}(Y|X)]+\mathrm{Var}(E(Y|X))
\end{aligned}
$$