Lecture 19: Multivariate transformation

In Chapter 2, we considered a single transformation of a single random variable.
We now consider a vector of transformations of a random vector.

The pdf of a multivariate transformation

Let X be a k-dimensional random vector with a joint pdf f_X and let $Y = g(X)$, where g is a Borel function from $\mathscr{R}^k$ to $\mathscr{R}^k$ (so that Y is a k-dimensional random vector). Let $A_1, ..., A_m$ be disjoint sets in $\mathscr{B}^k$ such that $P(X \in A_1 \cup \cdots \cup A_m) = 1$ and g on A_j is one-to-one with a nonvanishing Jacobian, i.e., the determinant $|\partial g(x)/\partial x| \neq 0$ on A_j, $j = 1, ..., m$. Then Y has the following joint pdf

$$f_Y(x) = \sum_{j=1}^{m} \Big| |\partial h_j(x)/\partial x| \Big| f_X\left(h_j(x)\right),$$

where h_j is the inverse function of g on A_j, and $\big||\partial h_j(x)/\partial x|\big|$ is the absolute value of the determinant of the matrix $\partial h_j(x)/\partial x$, $j = 1, ..., m$.

Example

Let $X = (X_1, X_2)$ be a 2-dimensional random vector with joint pdf f_X.
Consider first the transformation $u = g(x_1, x_2) = (u_1, u_2) = (x_1, x_1 + x_2)$.

$$\frac{\partial u}{\partial x} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \left|\frac{\partial u}{\partial x}\right| = 1 \qquad \left|\frac{\partial x}{\partial u}\right| = 1$$

Using the transformation theorem, the joint pdf of $U = g(X)$ is

$$f_U(u_1, u_2) = f_X(u_1, u_2 - u_1),$$

The marginal pdf of $U_2 = X_1 + X_2$ is then

$$f_{U_2}(u_2) = \int_{-\infty}^{\infty} f_X(u_1, u_2 - u_1) du_1.$$

The same result has been derived previously.

Next, consider the transformation $v = g(x_1, x_2) = (v_1, v_2) = (x_1/x_2, x_2)$.

$$\frac{\partial v}{\partial x} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{x_2} & -\frac{x_1}{x_2^2} \\ 0 & 1 \end{pmatrix} \qquad \left|\frac{\partial v}{\partial x}\right| = \frac{1}{x_2} \qquad \left|\frac{\partial x}{\partial v}\right| = v_2$$

Using the transformation theorem, the joint pdf of $V = g(X)$ is

$$f_V(v_1, v_2) = |v_2| f_X(v_1 v_2, v_2),$$

The marginal pdf of $V_1 = X_1/X_2$ is

$$f_{V_1}(v_1) = \int_{-\infty}^{\infty} |v_2| f_X(v_1 v_2, v_2) dv_2.$$

In particular, if X_1 and X_2 are independent, then

$$f_{V_1}(z) = \int_{-\infty}^{\infty} |v_2| f_{X_1}(v_1 v_2) f_{X_2}(v_2) dv_2$$

Finally, consider the transformation $w = g(x_1, x_2) = (w_1, w_2) = (x_1 x_2, x_2)$.

$$\frac{\partial w}{\partial x} = \begin{pmatrix} \frac{\partial w_1}{\partial x_1} & \frac{\partial w_1}{\partial x_2} \\ \frac{\partial w_2}{\partial x_1} & \frac{\partial w_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} x_2 & x_1 \\ 0 & 1 \end{pmatrix} \qquad \left| \frac{\partial w}{\partial x} \right| = x_2 \qquad \left| \frac{\partial x}{\partial w} \right| = \frac{1}{w_2}$$

Using the transformation theorem, the joint pdf of $W = g(X)$ is

$$f_W(w_1, w_2) = |w_2|^{-1} f_X(w_1 w_2^{-1}, w_2),$$

The marginal pdf of $W_1 = X_1 X_2$ is

$$f_{W_1}(w_1) = \int_{-\infty}^{\infty} |w_2|^{-1} f_X(w_1 w_2^{-1}, w_2) dw_2.$$

In particular, if X_1 and X_2 are independent, then

$$f_{W_1}(w_1) = \int_{-\infty}^{\infty} |w_2|^{-1} f_{X_1}(w_1 w_2^{-1}) f_{X_2}(w_2) dw_2$$

A number of results can be derived from these results.

Ratio of two normal random variables

If X_1 and X_2 are independent and both have the normal distribution $N(0,1)$, then, the pdf of $V_1 = X_1/X_2$ is

$$\begin{aligned} f_{V_1}(v_1) &= \int_{-\infty}^{\infty} |v_2| \frac{e^{-(v_1 v_2)^2/2}}{\sqrt{2\pi}} \frac{e^{-v_2^2/2}}{\sqrt{2\pi}} dv_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |v_2| e^{-(1+v_1^2)v_2^2/2} dv_2 \\ &= \frac{1}{\pi} \int_0^{\infty} v_2 e^{-(1+v_1^2)v_2^2/2} dv_2 = \frac{1}{\pi} \int_0^{\infty} e^{-(1+v_1^2)s} ds \\ &= \frac{1}{\pi(1+v_1^2)} \end{aligned}$$

which is the pdf of $Cauchy(0,1)$.

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&= \frac{1}{\pi} \int_0^{\infty} v_2 e^{-(1+v_1^2)v_2^2/2} dv_2 = \frac{1}{\pi} \int_0^{\infty} e^{-(1+v_1^2)s} ds \\
&= \frac{1}{\pi(1+v_1^2)}
\end{aligned}$$

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Ratio of chi-square random variables and F-distribution

Let X_1 and X_2 be independent random variables having the chi-square distributions with degrees of freedom n_1 and n_2, respectively.
By the transformation theorem, the p.d.f. of $Z = X_1/X_2$ is

$$\begin{aligned} f_Z(z) &= \int_0^\infty x_2 \frac{(zx_2)^{n_1/2-1}e^{-zx_2/2}}{2^{n_1/2}\Gamma(n_1/2)} \frac{x_2^{n_2/2-1}e^{-x_2/2}}{2^{n_2/2}\Gamma(n_2/2)} dx_2 \\ &= \frac{z^{n_1/2-1}}{2^{(n_1+n_2)/2}\Gamma(n_1/2)\Gamma(n_2/2)} \int_0^\infty x_2^{(n_1+n_2)/2-1} e^{-(1+z)x_2/2} dx_2 \\ &= \frac{\Gamma[(n_1+n_2)/2]}{\Gamma(n_1/2)\Gamma(n_2/2)} \frac{z^{n_1/2-1}}{(1+z)^{(n_1+n_2)/2}}, \qquad z > 0 \end{aligned}$$

where the last equality follows from the fact that

$$\frac{1}{2^{(n_1+n_2)/2}\Gamma[(n_1+n_2)/2]} x_2^{(n_1+n_2)/2-1} e^{-x_2/2} \qquad x_2 > 0$$

is the p.d.f. of the chi-square distribution with degrees of freedom $n_1 + n_2$.

Making another transformation $Y = (n_2/n_1)Z = (X_1/n_1)/(X_2/n_2)$ and applying the (univariate) transformation theorem, we obtain that the pdf of Y is

$$\frac{n_1^{n_1/2} n_2^{n_2/2} \Gamma[(n_1+n_2)/2]}{\Gamma(n_1/2)\Gamma(n_2/2)} \frac{y^{n_1/2-1}}{(n_1 z + n_2)^{(n_1+n_2)/2}}, \qquad y > 0$$

which is the pdf of the well-known F-distribution with degrees of freedom n_1 and n_2.

t-distribution

Let U_1 and U_2 be independent random variables, $U_1 \sim N(0,1)$ and U_2 has the chi-square distribution with degrees of freedom n.

What is the distribution of $T = U_1/\sqrt{U_2/n}$?

Let $X_1 = U_1^2$ and $X_2 = U_2$.

Then X_1 and X_2 are independent, because U_1 and U_2 are independent.

By a result obtained previously, X_1 has the chi-square distribution with degree of freedom 1.

From the previous proof, $Y = X_1/(X_2/n)$ has the F-distribution with degrees of freedom 1 and n.

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$$\frac{n_1^{n_1/2} n_2^{n_2/2} \Gamma[(n_1+n_2)/2]}{\Gamma(n_1/2)\Gamma(n_2/2)} \frac{y^{n_1/2-1}}{(n_1 z + n_2)^{(n_1+n_2)/2}}, \qquad y > 0$$

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t-distribution

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By a result obtained previously, X_1 has the chi-square distribution with degree of freedom 1.
From the previous proof, $Y = X_1/(X_2/n)$ has the F-distribution with degrees of freedom 1 and n.

The pdf of Y is

$$\frac{n^{n/2}\Gamma[(n+1)/2]y^{-1/2}}{\sqrt{\pi}\Gamma(n/2)(n+y)^{(n+1)/2}} \qquad y > 0$$

Applying the univariate transformation theorem, we obtain that the pdf of $W = \sqrt{Y}$ is ($dy = 2wdw$ when $y = w^2$)

$$\frac{2n^{n/2}\Gamma[(n+1)/2]}{\sqrt{\pi}\Gamma(n/2)(n+w^2)^{(n+1)/2}} \qquad w > 0$$

Note that

$$T = \begin{cases} W & U_1 \geq 0 \\ -W & U_1 < 0. \end{cases}$$

and

$$P(T < -t) = P(T > t). \qquad t > 0.$$

Hence the pdf of T is

$$\frac{n^{n/2}\Gamma[(n+1)/2]}{\sqrt{\pi}\Gamma(n/2)(n+t^2)^{(n+1)/2}} \qquad t \in \mathscr{R}$$

This is the pdf of the well-known t-distribution with degrees of freedom n.

Example 4.3.3 (product of two beta random variables)

Let $X \sim beta(\alpha, \beta)$ and $Y \sim beta(\alpha + \beta, \gamma)$ be independent so that the joint pdf is

$$f_{X,Y}(y,x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} y^{\alpha+\beta-1}(1-y)^{\gamma-1}$$

$$0 < x < 1, \quad 0 < y < 1$$

Consider $U = XY$ and $V = X$.

An application of the early result for product gives that

$$f_U(u) = \int_{-\infty}^{\infty} |v|^{-1} f(uv^{-1}, v) dv$$

But we need to be careful when the pdf is 0 in some intervals.

Note that the range of (U, V) is the region $0 < u < v < 1$ (why?).

Thus,

$$f_U(u) = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \int_u^1 v^{\alpha-1}(1-v)^{\beta-1} \left(\frac{u}{v}\right)^{\alpha+\beta-1} \left(1 - \frac{u}{v}\right)^{\gamma-1} \frac{1}{v} dv$$

for $0 < u < 1$ ($f_U(u) = 0$ when $u < 0$ or $u > 1$.)

Making the change of variable $s = (u/v - u)/(1-u)$ so that $dy = -u/[v^2(1-u)]$, $s = 0$ when $v = 1$ and $s = 1$ when $v = u$, we obtain that

$$\begin{aligned} f_U(u) &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \int_u^1 \left(\frac{u}{v} - u\right)^{\beta-1} \left(1 - \frac{u}{v}\right)^{\gamma-1} \frac{u}{v^2} dv \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \int_0^1 s^{\beta-1}(1-s)^{\gamma-1} ds \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \\ &= \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta+\gamma)} u^{\alpha-1}(1-u)^{\beta+\gamma-1} \qquad 0 < u < 1 \end{aligned}$$

Thus, the product $XY \sim beta(\alpha, \beta+\gamma)$.

Example

If $X_i \sim N(0,1)$, $i = 1,...,n$, are independent, we want to show that, for every $i = 1,...,n$, the random variables $U = X_1^2 + \cdots + X_n^2$ and $V_i = X_i^2/U$ are independent, and we want to find the pdf of V_i.

The joint pdf of $X_1, ..., X_n$ is

$$f_X(x_1, ..., x_n) = \frac{1}{(\sqrt{2\pi})^n} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} x_i^2\right) \qquad x_i \in \mathscr{R},\ i = 1, ..., n$$

For $Y_i = X_i^2$, $i = 1, ..., n$, an application of the transformation theorem shows that the joint pdf of $Y_1, ..., Y_n$ is

$$f_Y(y_1, ..., y_n) = \frac{1}{2^{n/2}\sqrt{y_1 \cdots y_n}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} y_i\right) \qquad y_i > 0,\ i = 1, ..., n.$$

Let $U = Y_1 + \cdots + Y_n$ and $V_i = Y_i/U$, $i = 1, ..., n$.

Then $Y_i = UV_i$, $i = 1, ..., n-1$, and $Y_n = UV_n = U(1 - V_1 - \cdots - V_{n-1})$.

$$\left|\frac{\partial(y_1, ..., y_n)}{\partial(u, v_1, ..., v_{n-1})}\right| = \begin{vmatrix} v_1 & u & 0 & \cdots & 0 \\ v_2 & 0 & u & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_{n-1} & 0 & 0 & \cdots & u \\ v_n & 0 & 0 & \cdots & 0 \end{vmatrix} = v_n u^{n-1}$$

where $v_n = 1 - v_1 - \cdots - v_{n-1}$.

The joint pdf for U and $V_1, ..., V_{n-1}$ is

$$f_Y(uv_1, ..., uv_n) v_n u^{n-1} = \frac{1}{2^{n/2}} u^{n/2-1} e^{-u/2} \sqrt{\frac{1 - v_1 - \cdots - v_{n-1}}{v_1 \cdots v_{n-1}}}$$

Therefore, we know that U and $(V_1, ..., V_{n-1})$ are independent and, hence, U and V_i are independent for any $i = 1, ..., n-1$.

Since $V_n = 1 - V_1 - \cdots - V_{n-1}$, U and V_n are independent.

If $W = U - X_i^2$, then

$$V_i = \frac{X_i^2}{U} = \frac{X_i^2}{W + X_i^2} = \frac{X_i^2/W}{1 + X_i^2/W}$$

Previously we showed that $Z = X_i^2/W$ has pdf

$$\frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} \frac{z^{-1/2}}{(1+z)^{(n-1)/2}}, \qquad z > 0$$

Then the transformation $V_i = Z/(1+Z)$ gives the following pdf for V_i:

$$\frac{\Gamma(n/2)}{\Gamma(1/2)\Gamma((n-1)/2)} v^{-1/2} (1-v)^{(n-3)/2}, \qquad 0 < v < 1$$

which is *beta*$(1/2, (n-1)/2)$.