

Lecture 11: Uniformly most power tests

We first show some examples of applying the Neyman-Pearson lemma.

Example 8.3.14.

Let $X \sim \text{binomial}(2, \theta)$ with $\theta = \frac{1}{2}$ or $\frac{3}{4}$, $\Theta_0 = \{\frac{1}{2}\}$.

Note that

$$\frac{f_{\frac{3}{4}}(0)}{f_{\frac{1}{2}}(0)} = \frac{1}{4}, \quad \frac{f_{\frac{3}{4}}(1)}{f_{\frac{1}{2}}(1)} = \frac{3}{4}, \quad \frac{f_{\frac{3}{4}}(2)}{f_{\frac{1}{2}}(2)} = \frac{9}{4}$$

If we choose $\frac{3}{4} < c < \frac{9}{4}$, the Neyman-Pearson Lemma says that the test rejecting H_0 iff $X = 2$ is the UMP test of size $\alpha_c = P_{\frac{1}{2}}(X = 2) = \frac{1}{4}$.

If we choose $\frac{1}{4} < c < \frac{3}{4}$, the Neyman-Pearson Lemma says that the test rejecting H_0 iff $X = 1$ or 2 is the UMP test of size $\alpha_c = P_{\frac{1}{2}}(X \geq 1) = \frac{3}{4}$.

Choosing $c < \frac{1}{4}$ or $c > \frac{9}{4}$ yields the UMP test of size 1 or 0.

What happens if we choose $c = \frac{3}{4}$?

If $c = \frac{3}{4}$, Theorem 8.3.12 tells us that the UMP test rejects H_0 when $X = 2$ and accepts H_0 when $X = 0$, but leaves our action undetermined on $\{X = 1\} = \{f_{\frac{3}{4}}(X) = cf_{\frac{1}{2}}(X)\}$, an event with a positive probability.

When $X = 1$, we must either reject (then we get the UMP test of size $\frac{3}{4}$ as above) or accept (then we get the UMP test of size $\frac{1}{4}$ as above).

Example.

This example shows that the event $\{X : f_{\theta_1}(x) = cf_{\theta_0}(X)\}$ may have a positive probability even when X is continuous.

Let $X \sim \text{uniform}(0, \theta)$ with $\theta = \theta_0$ or θ_1 ($\theta_0 < \theta_1$), $\Theta_0 = \{\theta_0\}$.

$$f_{\theta_0}(x) = \begin{cases} \frac{1}{\theta_0} & 0 < x < \theta_0 \\ 0 & \text{otherwise} \end{cases} \quad f_{\theta_1}(x) = \begin{cases} \frac{1}{\theta_1} & 0 < x < \theta_1 \\ 0 & \text{otherwise} \end{cases}$$

If $c < \theta_0/\theta_1$, then

$$\{x : f_{\theta_1}(x) > cf_{\theta_0}(x)\} = (0, \theta_1)$$

and the UMP test always rejects H_0 and has size 1 (not desirable).

If we choose $c > \theta_0/\theta_1$, then

$$\{x : f_{\theta_1}(x) > cf_{\theta_0}(x)\} = [\theta_0, \theta_1) \quad \{x : f_{\theta_1}(x) < cf_{\theta_0}(x)\} = (0, \theta_0)$$

$$\{x : f_{\theta_1}(x) = cf_{\theta_0}(x)\} = C \quad \text{with } P_{\theta_0}(C) = P_{\theta_1}(C) = 0$$

Hence the size of the UMP test is

$$P_{\theta_0}(f_{\theta_1}(X) > cf_{\theta_0}(X)) = P_{\theta_0}(\theta_0 \leq X < \theta_1) = 0$$

This is not desirable either.

Choosing $c = \theta_0/\theta_1$ leads to

$$\{x : f_{\theta_1}(x) > cf_{\theta_0}(x)\} = [\theta_0, \theta_1) \quad \{x : f_{\theta_1}(x) = cf_{\theta_0}(x)\} = (0, \theta_0)$$

$$\{x : f_{\theta_1}(x) < cf_{\theta_0}(x)\} = C \quad \text{with } P_{\theta_0}(C) = P_{\theta_1}(C) = 0$$

Now, we must specify which part of $\{x : f_{\theta_1}(x) = cf_{\theta_0}(x)\} = (0, \theta_0)$ is in the rejection region R .

If we put the entire $(0, \theta_0)$ into R^c (or R), the size of the UMP is 0 (or 1).

Thus, we should put part of $(0, \theta_0)$ into R .

Let α be a desired level.

Choosing $R = [\theta_0, \theta_1) \cup [\theta_0 - \alpha\theta_0, \theta_0) = [\theta_0 - \alpha\theta_0, \theta_1)$ gives

$$\text{the size of the UMP test} = P_{\theta_0}(\theta_0 - \alpha\theta_0 \leq X < \theta_1) = \alpha$$

If X is continuous, and $f_{\theta_0}(x)$ and $f_{\theta_1}(x)$ have the same support, then the event $\{f_{\theta_1}(X) = cf_{\theta_0}(X)\}$ is typically of probability 0.

Example 8.3.15.

Let $X_1, \dots, X_n$ be iid from $N(\theta, \sigma^2)$ with known σ^2 and unknown $\theta = \theta_0$ or θ_1 , $\theta_0 > \theta_1$, and $\Theta_0 = \{\theta_0\}$.

The sample mean $\bar{X}$ is sufficient for θ .

By Corollary 8.3.13, the UMP test has rejection region

$$\frac{\exp(-(\bar{x} - \theta_1)^2/(2\sigma^2))}{\exp(-(\bar{x} - \theta_0)^2/(2\sigma^2))} > c$$

which is equivalent to (under the fact that $\theta_0 > \theta_1$)

$$\bar{x} < \frac{(2\sigma^2 \log c)/n - \theta_0^2 + \theta_1^2}{2(\theta_1 - \theta_0)}$$

The right hand side increases from $-\infty$ to ∞ as c increases from 0 to ∞ . Thus, for any given α , we can choose a c so that the rejection region is $\{\bar{x} < \theta_0 - \sigma z_\alpha / \sqrt{n}\}$, which leads to a UMP test of size

$$P_{\theta_0}(\bar{X} < \theta_0 - \sigma z_\alpha / \sqrt{n}) = \alpha.$$

An interesting phenomenon in Example 8.3.15 is that the UMP test of size α (for a given α) has rejection region $R = \{\bar{X} < \theta_0 - \sigma z_\alpha / \sqrt{n}\}$ that does not depend on θ_1 ; it depends on θ_0 and α .

In such a case, the test with rejection region R is in fact a UMP test of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$.

Another observation from Example 8.3.15 is that

$$P_\theta(X \in R) = P_\theta(\bar{X} < \theta_0 - \sigma z_\alpha / \sqrt{n}) = \Phi\left(\frac{\sqrt{n}(\theta_0 - \theta)}{\sigma} - z_\alpha\right)$$

is decreasing in θ and hence

$$\sup_{\theta \geq \theta_0} P_\theta(X \in R) = \Phi(-z_\alpha) = \alpha$$

It means that the test with rejection region R is in fact a UMP test of size α for testing $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$.

This gives us a way to extend the result in Theorem 8.3.12 to hypotheses that are not simple versus simple.

Formally we have the following result.

Lemma A.

Let $\Theta_0 \subset \Theta$ be a subset containing θ_0 . Suppose that there is a test T_* of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ for every $\theta_1 \in \Theta_0^c$ and the rejection region R of T_* does not depend on θ_1 .

- ① T_* is a UMP test of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \in \Theta_0^c$.
- ② If $P_\theta(X \in R) \leq P_{\theta_0}(X \in R)$ for all $\theta \in \Theta_0$, then T_* is a UMP test of size α for testing $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_0^c$.

Proof.

First, T_* is a test since it does not depend on θ_1 .

- ① Consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \in \Theta_0^c$.
For any test T of level α , T is also of level α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ with any $\theta_1 \in \Theta_0^c$ and, hence, $\beta_{T_*}(\theta_1) \geq \beta_T(\theta_1)$. Since θ_1 is arbitrary, this proves that T_* is a UMP test of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \in \Theta_0^c$.
- ② This follows from the result in the previous part, $\theta_0 \in \Theta_0$, and

$$\sup_{\theta \in \Theta_0} P_\theta(X \in R) \leq P_{\theta_0}(X \in R) = \alpha$$

In the previous discussion, the following are the key conditions for the success of extending the result in Theorem 8.3.12 to a not simple versus simple testing problem.

- The UMP test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ does not depend on θ_1 .
- The parameter θ is univariate and the hypothesis H_0 is of the form $\theta \geq \theta_0$ or $\theta \leq \theta_0$; this type of hypotheses are called one-sided hypotheses.
- The type I error probability reaches its maximum at θ_0 , which is implied by the property that the power function is monotone.

A class of problems having these conditions corresponds to families having the so called monotone likelihood ratio (MLR) property.

Definition 8.3.16.

A family of pdf's or pmf's $\{f_\theta(x) : \theta \in \Theta\}$ for a sample X with $\Theta \subset \mathcal{R}$ has a monotone likelihood ratio (MLR) in $T(X)$ (a real valued statistic) if, for every $\theta_2 > \theta_1$, $\theta_j \in \Theta$, $j = 1, 2$, $f_{\theta_2}(x)/f_{\theta_1}(x)$ is a nondecreasing function of $T(x)$ on the set $\{t : f_{\theta_1}(x) > 0 \text{ or } f_{\theta_2}(x) > 0\}$. ($c/0 = \infty$ if $0 < c < \infty$.)

Many families of distributions have an MLR; for example,

- if $f_{\theta}(x) = h(x)c(\theta)e^{w(\theta)T(x)}$ (one parameter exponential family), then the family has an MLR in T if $w(\theta)$ is monotone;
- the *exponential* (θ, c) with a known c ;
- the *uniform* $(0, \theta)$;

An example of a family that does not have MLR is the *Cauchy* (θ, c) with a known c .

Example.

Let $X_1, \dots, X_n$ be iid from *uniform* $(0, \theta)$ with unknown $\theta > 0$.

The joint pdf of $X = (X_1, \dots, X_n)$ is $f_{\theta}(x) = \theta^{-n} I_{(0, \theta)}(x_{(n)})$, where $x_{(n)}$ is the value of $X_{(n)}$ and $I_{(0, \theta)}(x_{(n)})$ is an indicator function.

For $\theta_1 < \theta_2$,

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \frac{\theta_1^n I_{(0, \theta_2)}(x_{(n)})}{\theta_2^n I_{(0, \theta_1)}(x_{(n)})},$$

which is a nondecreasing function of $x_{(n)}$ for x 's at which at least one of $f_{\theta_1}(x)$ and $f_{\theta_2}(x)$ is positive, i.e., $x_{(n)} < \theta_2$.

Hence the family of distributions of X has MLR in $X_{(n)}$.

Lemma B.

Suppose that the pdf or pmf of X is in a family (indexed by a real-valued θ) with MLR in $T(X)$. If $\psi(t)$ is a nondecreasing function of t , then $g(\theta) = E_{\theta}[\psi(T)]$ is a nondecreasing function of θ .

The following is the main result for a UMP test in one-sided hypothesis problems.

Theorem 8.3.17 (Karlin-Rubin).

Let X be a sample with pdf or pmf in a family indexed by $\theta \in \Theta \subset \mathcal{R}$. Suppose that the family has MLR in T . Let θ_0 be a given constant. For any $c \in \mathcal{R}$, the test rejecting H_0 iff $T > c$ is a UMP test of size $\alpha_c = P_{\theta_0}(T > c)$ for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$; and the test rejecting H_0 iff $T < c$ is a UMP test of size $\alpha_c = P_{\theta_0}(T < c)$ for testing $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$.

Proof.

We prove the case for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$. Consider the hypotheses $\theta = \theta_0$ versus $\theta = \theta_1$ with any $\theta_1 > \theta_0$.

A UMP test is given in Theorem 8.3.12 with rejection region

$$\{x : f_{\theta_1}(x) > \tilde{c}f_{\theta_0}(x)\}$$

where $f_{\theta}(x)$ is the joint pdf or pmf of X .

Since the family has MLR in T , i.e., $f_{\theta_1}(x)/f_{\theta_0}(x)$ is a nondecreasing function of $T(x)$, this rejection region is the same as $\{x : T(x) > c\}$ with a c satisfying $\beta_T(\theta_0) = P_{\theta_0}(T > c) = \alpha_c$.

Since this region does not depend on θ_1 , it follows from Lemma A that this is a UMP test of size α_c for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta > \theta_0$.

To complete the proof, as we argued earlier, we only need to show that the power function of the test with rejection region $\{x : T(x) > c\}$ is nondecreasing in θ .

Let $\psi(t) = I_{(c, \infty)}(t)$, the indicator function of the interval (c, ∞) .

Then Lemma B implies that $\beta(\theta) = E_{\theta}[\psi(T)] = P_{\theta}(T > c)$ is nondecreasing in θ for any fixed c .

This completes the proof.

In view of Corollary 8.3.13, if there is a sufficient statistic $T(X)$, then we can apply Theorem 8.3.17 with the sample X replaced by T (treating T as the observation).

Often, the family of pdf's or pmf's for T has MLR in T .

The following are some examples.

Example 8.3.18.

Let $X_1, \dots, X_n$ be iid from $N(\theta, \sigma^2)$ with known σ^2 and unknown $\theta \in \mathcal{R}$.

Consider testing $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$ with a constant θ_0 .

Since $\bar{X} \sim N(\theta, \sigma^2/n)$ is sufficient for θ and the family $N(\theta, \sigma^2/n)$ has MLR in $\bar{X}$, an application of Theorem 8.3.17 shows that a UMP test of size α has the rejection region $\{\bar{X} < \theta_0 - \sigma z_\alpha / \sqrt{n}\}$.

This test is the same as that derived in Example 8.3.15.

Example 8.2.3.

Let $X_1, \dots, X_n$ be iid from *exponential* $(\theta, 1)$ with unknown $\theta \in \mathcal{R}$ and hypotheses be $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ for a fixed constant θ_0 .

We previously showed that the LRT having rejection region

$$\{X_{(1)} \geq \theta_0 - n^{-1} \log \alpha\}$$

is a size α test.

Note that $X_{(1)}$ is sufficient for θ and has pdf

$$f_{\theta}(t) = \begin{cases} ne^{-n(t-\theta)} & t \geq \theta \\ 0 & t < \theta \end{cases}$$

For $\theta_1 < \theta_2$,

$$\frac{f_{\theta_2}(t)}{f_{\theta_1}(t)} = \begin{cases} 0 & \theta_1 \leq t < \theta_2 \\ e^{n(\theta_2 - \theta_1)} & \theta_2 \leq t \end{cases}$$

is nondecreasing in t .

Hence, Theorem 8.3.17 applies and the LRT is a UMP test of size α .

Example.

Let $X_1, \dots, X_n$ be iid from *uniform*(0, θ), $\theta > 0$.

Consider the hypotheses $H_0 : \theta \leq \theta_0$ and $H_1 : \theta > \theta_0$.

We have shown that the pdf of $X_1, \dots, X_n$ is in a family with MLR in $X_{(n)}$.

We previously showed that the LRT having rejection region

$$\{X_{(1)} \geq \theta_0 - n^{-1} \log \alpha\}$$

is a size α test.

Note that $X_{(1)}$ is sufficient for θ and has pdf

$$f_{\theta}(t) = \begin{cases} ne^{-n(t-\theta)} & t \geq \theta \\ 0 & t < \theta \end{cases}$$

For $\theta_1 < \theta_2$,

$$\frac{f_{\theta_2}(t)}{f_{\theta_1}(t)} = \begin{cases} 0 & \theta_1 \leq t < \theta_2 \\ e^{n(\theta_2 - \theta_1)} & \theta_2 \leq t \end{cases}$$

is nondecreasing in t .

Hence, Theorem 8.3.17 applies and the LRT is a UMP test of size α .

Example.

Let $X_1, \dots, X_n$ be iid from $uniform(0, \theta)$, $\theta > 0$.

Consider the hypotheses $H_0 : \theta \leq \theta_0$ and $H_1 : \theta > \theta_0$.

We have shown that the pdf of $X_1, \dots, X_n$ is in a family with MLR in $X_{(n)}$.

By Theorem 8.3.17, a UMP test of size α rejects H_0 when $X_{(n)} > c$. Since $X_{(n)}$ has pdf $n\theta^{-n}x^{n-1}I_{(0,\theta)}(x)$,

$$\alpha = \beta(\theta_0) = \frac{n}{\theta_0^n} \int_c^{\theta_0} x^{n-1} dx = 1 - \frac{c^n}{\theta_0^n}$$

Hence $c = \theta_0(1 - \alpha)^{1/n}$.

The power function of this test when $\theta > \theta_0$ is

$$\beta(\theta) = \frac{n}{\theta^n} \int_c^{\theta} x^{n-1} dx = 1 - \frac{\theta_0^n(1 - \alpha)}{\theta^n}$$

Example.

Let $X_1, \dots, X_n$ be iid from $Poisson(\theta)$ with unknown $\theta > 0$.

The pmf of $X = (X_1, \dots, X_n)$ is in a one-parameter exponential family with $T(X) = \sum_{i=1}^n X_i \sim Poisson(n\theta)$.

By Theorem 8.3.17, a UMP test of size α_c for $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ has rejection region $\{T > c\}$ with

$$\alpha_c = P_{\theta_0}(T > c) = \sum_{j=c+1}^{\infty} \frac{e^{-n\theta_0} (n\theta_0)^j}{j!}$$

Existence of a UMP without monotone likelihood ratio

Let $(X_1, \dots, X_n)$, $n \geq 2$, be a random sample from $\text{uniform}(\theta, \theta + 1)$, $\theta \in \mathcal{R}$.

It can be shown that this family does not have MLR.

For testing

$$H_0 : \theta \leq 0 \quad \text{versus} \quad H_1 : \theta > 0,$$

we want to show that, for any $\alpha \in (0, 1)$, the test T_* accepts H_0 iff $X_{(1)} < 1 - \alpha^{1/n}$ and $X_{(n)} < 1$ is a UMP test of size α .

We can consider tests that are functions of the sufficient statistic $(X_{(1)}, X_{(n)})$.

Since the pdf of $(X_{(1)}, X_{(n)})$ is

$$f_{\theta}(x, y) = n(n-1)(y-x)^{n-2} \quad \theta < x < y < \theta + 1$$

a direct calculation shows that the power function of T_* is

$$\beta_{T_*}(\theta) = \begin{cases} 0 & \theta < -\alpha^{1/n} \\ (\theta + \alpha^{1/n})^n & -\alpha^{1/n} \leq \theta \leq 0 \\ 1 + \alpha - (1 - \theta)^n & 0 < \theta \leq 1 - \alpha^{1/n} \\ 1 & \theta > 1 - \alpha^{1/n}. \end{cases}$$

Hence the size of T_* is α .

Let $\beta_T(\theta)$ be the power function of another test $T = T(X_{(1)}, X_{(n)})$.

Since $\beta_{T_*}(\theta) = 1$ when $\theta > 1 - \alpha^{1/n}$, it suffices to show that

$$\beta_{T_*}(\theta) \geq \beta_T(\theta)$$

for $\theta \in (0, 1 - \alpha^{1/n})$.

Define

$$A = \{0 < X_{(1)} \leq X_{(n)} < 1\}$$

$$B = \{\theta < X_{(1)} \leq X_{(n)} < 1\}$$

$$C = \{1 < X_{(1)} \leq X_{(n)} < \theta + 1\}$$

Let $T_*(X_{(1)}, X_{(n)})$ and $T(X_{(1)}, X_{(n)})$ be the indicators of the rejection regions of T_* and T , respectively, and

$$h(x, y) = n(n-1)(y-x)^{n-2}$$

If $\theta \in (0, 1 - \alpha^{1/n})$, then

$$\begin{aligned}
& \beta_{T_*}(\theta) - \beta_T(\theta) \\
&= \int_B (T_* - T)(x, y) h(x, y) dx dy + \int_C (T_* - T)(x, y) h(x, y) dx dy \\
&= \int_B (T_* - T)(x, y) h(x, y) dx dy + \int_C (1 - T)(x, y) h(x, y) dx dy \\
&\geq \int_B T_*(x, y) h(x, y) dx dy - \int_B T(x, y) h(x, y) dx dy \\
&\geq \int_A T_*(x, y) h(x, y) dx dy - \int_B T(x, y) h(x, y) dx dy \\
&\geq \int_A T_*(x, y) h(x, y) dx dy - \int_A T(x, y) h(x, y) dx dy \\
&= \beta_{T_*}(0) - \beta_T(0) \\
&= \alpha - \beta_T(0),
\end{aligned}$$

where the second equality follows from $T_* = 1$ when $(X_{(1)}, X_{(n)}) \in C$, the third equality follows from $T_* = 0$ when $(X_{(1)}, X_{(n)}) \in A$ but $(X_{(1)}, X_{(n)}) \notin B$ (since $0 < X_{(1)} \leq \theta \leq 1 - \alpha^{1/n}$), and the second inequality follows from $B \subset A$.

Therefore, if T has level α , then $\beta_{T_*}(\theta) \geq \beta_T(\theta)$ for all $\theta \in (0, 1 - \alpha^{1/n})$.