Lecture 3: Minimaxity and admissibility

Consider estimators of a real-valued $\vartheta = g(\theta)$ based on a sample X from P_θ, $\theta \in \Theta$, under loss L and risk $R_T(\theta) = E[L(T(X), \theta)]$.

Minimax estimator

An estimator δ is minimax if $\sup_\theta R_\delta(\theta) = \inf_{\text{all } T} \sup_\theta R_T(\theta)$

Discussion

- A minimax estimator can be very conservative and unsatisfactory. It tries to do as well as possible in the worst case.
- A unique minimax estimator is admissible, since any estimator better than a minimax estimator is also minimax.
- We should find an admissible minimax estimator.
- Different for UMVUE: if a UMVUE is inadmissible, it is dominated by a biased estimator)
- If a minimax estimator has some other good properties (e.g., it is a Bayes estimator), then it is often a reasonable estimator.

Minimax estimator

The following result shows when a Bayes estimator is minimax.

Theorem 4.11 (minimaxity of a Bayes estimator)

Let Π be a proper prior on Θ and δ be a Bayes estimator of ϑ w.r.t. Π.
Suppose δ has constant risk on Θ_Π.
If $\Pi(\Theta_\Pi) = 1$, then δ is minimax.
If, in addition, δ is the unique Bayes estimator w.r.t. Π, then it is the unique minimax estimator.

Proof

Let T be any other estimator of ϑ. Then

$$\sup_{\theta \in \Theta} R_T(\theta) \geq \int_{\Theta_\Pi} R_T(\theta) d\Pi \geq \int_{\Theta_\Pi} R_\delta(\theta) d\Pi = \sup_{\theta \in \Theta} R_\delta(\theta).$$

If δ is the unique Bayes estimator, then the second inequality in the previous expression should be replaced by $>$ and, therefore, δ is the unique minimax estimator.

Example 4.18

Let $X_1,...,X_n$ be i.i.d. binary random variables with $P(X_1 = 1) = p$.
Consider the estimation of p under the squared error loss.
The UMVUE $\bar{X}$ has risk $p(1-p)/n$ which is not constant.
In fact, $\bar{X}$ is not minimax (Exercise 67).
To find a minimax estimator by applying Theorem 4.11, we consider the Bayes estimator w.r.t. the beta distribution $B(\alpha,\beta)$ with known α and β (Exercise 1):

$$\delta(X) = (\alpha + n\bar{X})/(\alpha+\beta+n).$$

$$R_\delta(p) = [np(1-p) + (\alpha - \alpha p - \beta p)^2]/(\alpha+\beta+n)^2.$$

To apply Theorem 4.11, we need to find values of $\alpha > 0$ and $\beta > 0$ such that $R_\delta(p)$ is constant.
It can be shown that $R_\delta(p)$ is constant if and only if $\alpha = \beta = \sqrt{n}/2$, which leads to the unique minimax estimator

$$T(X) = (n\bar{X} + \sqrt{n}/2)/(n+\sqrt{n}).$$

The risk of T is $R_T = 1/[4(1+\sqrt{n})^2]$.

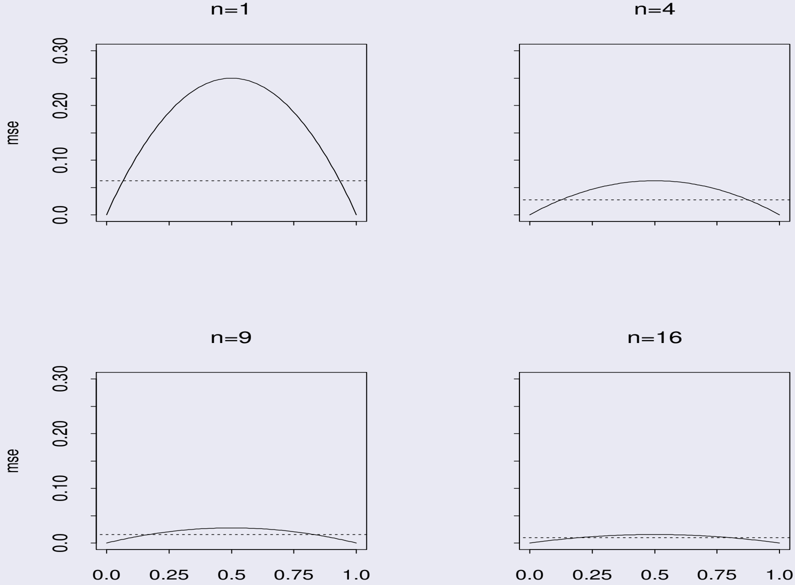
Example 4.18 (continued)

Note that T is a Bayes estimator and has some good properties.
Comparing the risk of T with that of $\bar{X}$, we find that T has smaller risk if and only if

$$p \in \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{n}{(1+\sqrt{n})^2}},\ \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{n}{(1+\sqrt{n})^2}} \right).$$

Thus, for a small n, T is better (and can be much better) than $\bar{X}$ for most of the range of p (Figure 4.1).
When $n \to \infty$, the above interval shrinks toward $\frac{1}{2}$.
Hence, for a large (and even moderate) n, $\bar{X}$ is better than T for most of the range of p (Figure 4.1).
The limit of the asymptotic relative efficiency of T w.r.t. $\bar{X}$ is $4p(1-p)$, which is always smaller than 1 when $p \neq \frac{1}{2}$ and equals 1 when $p = \frac{1}{2}$.
Minimaxity depends strongly on the loss function.
Under the loss function $L(p,a) = (a-p)^2/[p(1-p)]$, $\bar{X}$ has constant risk and is the unique Bayes estimator w.r.t. the uniform prior on $(0,1)$.
By Theorem 4.11, $\bar{X}$ is the unique minimax estimator.
The risk, however, of T is $1/[4(1+\sqrt{n})^2 p(1-p)]$, which is unbounded.

Figure 4.1. mse's of $\bar{X}$ (curve) and $T(X)$ (straight line) in Example 4.18



How to find a minimax estimator?

Candidates for minimax: estimators having constant risks.

Theorem 4.11 (minimaxity of a Bayes estimator)

A limit of Bayes estimators

In many cases a constant risk estimator is not a Bayes estimator (e.g., an unbiased estimator under the squared error loss), but a limit of Bayes estimators w.r.t. a sequence of priors.
The next result may be used to find a minimax estimator.

Theorem 4.12

Let Π_j, $j = 1, 2, ...$, be a sequence of priors and r_j be the Bayes risk of a Bayes estimator of ϑ w.r.t. Π_j.
Let T be a constant risk estimator of ϑ.
If $\liminf_j r_j \geq R_T$, then T is minimax.

Although Theorem 4.12 is more general than Theorem 4.11 in finding minimax estimators, it does not provide uniqueness of the minimax estimator even when there is a unique Bayes estimator w.r.t. each Π_j.

Example 2.25

Let $X_1, ..., X_n$ be i.i.d. components having the $N(\mu, \sigma^2)$ distribution with an unknown $\mu = \theta \in \mathscr{R}$ and a known σ^2.
If the prior is $N(\mu_0, \sigma_0^2)$, then the posterior of θ given $X = x$ is $N(\mu_*(x), c^2)$ with

$$\mu_*(x) = \frac{\sigma^2}{n\sigma_0^2 + \sigma^2}\mu_0 + \frac{n\sigma_0^2}{n\sigma_0^2 + \sigma^2}\bar{x} \qquad \text{and} \qquad c^2 = \frac{\sigma_0^2\sigma^2}{n\sigma_0^2 + \sigma^2}$$

We now show that $\bar{X}$ is minimax under the squared error loss.
For any decision rule T,

$$\sup_{\theta \in \mathscr{R}} R_T(\theta) \geq \int_{\mathscr{R}} R_T(\theta) d\Pi(\theta) \geq \int_{\mathscr{R}} R_{\mu_*}(\theta) d\Pi(\theta)$$

$$= E\left\{[\vec{\theta} - \mu_*(X)]^2\right\} = E\left\{E\{[\vec{\theta} - \mu_*(X)]^2 | X\}\right\} = E(c^2) = c^2.$$

Since this result is true for any $\sigma_0^2 > 0$ and $c^2 \to \sigma^2/n$ as $\sigma_0^2 \to \infty$,

$$\sup_{\theta \in \mathscr{R}} R_T(\theta) \geq \frac{\sigma^2}{n} = \sup_{\theta \in \mathscr{R}} R_{\bar{X}}(\theta),$$

Example 2.25 (continued)

where the equality holds because the risk of $\bar{X}$ under the squared error loss is σ^2/n and independent of $\theta = \mu$.
Thus, $\bar{X}$ is minimax.

To discuss the minimaxity of $\bar{X}$ in the case where σ^2 is unknown, we need the following lemma.

Lemma 4.3

Let Θ_0 be a subset of Θ and T be a minimax estimator of ϑ when Θ_0 is the parameter space. Then T is a minimax estimator if

$$\sup_{\theta \in \Theta} R_T(\theta) = \sup_{\theta \in \Theta_0} R_T(\theta).$$

Proof

If there is an estimator T_0 with $\sup_{\theta \in \Theta} R_{T_0}(\theta) < \sup_{\theta \in \Theta} R_T(\theta)$, then

$$\sup_{\theta \in \Theta_0} R_{T_0}(\theta) \leq \sup_{\theta \in \Theta} R_{T_0}(\theta) < \sup_{\theta \in \Theta} R_T(\theta) = \sup_{\theta \in \Theta_0} R_T(\theta),$$

which contradicts the minimaxity of T when Θ_0 is the parameter space.
Hence, T is minimax when Θ is the parameter space.

Example 4.19

Let $X_1, ..., X_n$ be i.i.d. from $N(\mu, \sigma^2)$ with unknown $\theta = (\mu, \sigma^2)$.
Consider the estimation of μ under the squared error loss.
Suppose first that $\Theta = \mathscr{R} \times (0, c]$ with a constant $c > 0$.
Let $\Theta_0 = \mathscr{R} \times \{c\}$.
From Example 2.25, $\bar{X}$ is a minimax estimator of μ when the parameter space is Θ_0.
By Lemma 4.3, $\bar{X}$ is also minimax when the parameter space is Θ.
Although σ^2 is assumed to be bounded by c, the minimax estimator $\bar{X}$ does not depend on c.
Consider next the case where $\Theta = \mathscr{R} \times (0, \infty)$, i.e., σ^2 is unbounded.
Let T be any estimator of μ. For any fixed σ^2,

$$\frac{\sigma^2}{n} \leq \sup_{\mu \in \mathscr{R}} R_T(\theta),$$

since σ^2/n is the risk of $\bar{X}$ that is minimax when σ^2 is known.
Letting $\sigma^2 \to \infty$, we obtain that $\sup_\theta R_T(\theta) = \infty$ for any estimator T.
Thus, minimaxity is meaningless (any estimator is minimax).

Admissibility

The following is another result to show admissibility.

Theorem 4.14 (Admissibility in one-parameter exponential families)

Suppose that X has the p.d.f. $c(\theta)e^{\theta T(x)}$ w.r.t. a σ-finite measure ν, where $T(x)$ is real-valued and $\theta \in (\theta_-, \theta_+) \subset \mathscr{R}$.
Consider the estimation of $\vartheta = E[T(X)]$ under the squared error loss.
Let $\lambda \geq 0$ and γ be known constants and let

$$T_{\lambda,\gamma}(X) = (T + \gamma\lambda)/(1+\lambda).$$

Then a sufficient condition for the admissibility of $T_{\lambda,\gamma}$ is that

$$\int_{\theta_0}^{\theta_+} \frac{e^{-\gamma\lambda\theta}}{[c(\theta)]^{\lambda}} d\theta = \int_{\theta_-}^{\theta_0} \frac{e^{-\gamma\lambda\theta}}{[c(\theta)]^{\lambda}} d\theta = \infty,$$

where $\theta_0 \in (\theta_-, \theta_+)$.

Remarks

- Theorem 4.14 provides a class of admissible estimators.
- The reason why $T_{\lambda,\gamma}$ is considered is that it is often a Bayes estimator w.r.t. some prior; see Examples 2.25, 4.1, 4.7, and 4.8.
- Using this theorem and Theorem 4.13, we can obtain a class of minimax estimators.
- Although the proof of this theorem is more complicated than that of Theorem 4.3, the application of Theorem 4.14 is typically easier.
- To find minimax estimators, we may use the following result.

Corollary 4.3

Assume that X has the p.d.f. as described in Theorem 4.14 with $\theta_- = -\infty$ and $\theta_+ = \infty$.
(i) As an estimator of $\vartheta = E(T)$, $T(X)$ is admissible under the squared error loss and the loss $(a-\vartheta)^2/\operatorname{Var}(T)$.
(ii) T is the unique minimax estimator of ϑ under the loss $(a-\vartheta)^2/\operatorname{Var}(T)$.

Example 4.20

Let $X_1, ..., X_n$ be i.i.d. from $N(0, \sigma^2)$ with an unknown $\sigma^2 > 0$ and let $Y = \sum_{i=1}^n X_i^2$.
Consider the estimation of σ^2.
The risk of $Y/(n+2)$ is a constant under the loss $(a-\sigma^2)^2/\sigma^4$.
We now apply Theorem 4.14 to show that $Y/(n+2)$ is admissible.
Note that the joint p.d.f. of X_i's is of the form $c(\theta)e^{\theta T(x)}$ with $\theta = -n/(4\sigma^2)$, $c(\theta) = (-2\theta/n)^{n/2}$, $T(X) = 2Y/n$, $\theta_- = -\infty$, and $\theta_+ = 0$.
By Theorem 4.14, $T_{\lambda,\gamma} = (T + \gamma\lambda)/(1+\lambda)$ is admissible under the squared error loss if, for some $c > 0$,

$$\int_{-\infty}^{-c} e^{-\gamma\lambda\theta} \left(\frac{-2\theta}{n}\right)^{-n\lambda/2} d\theta = \int_0^c e^{\gamma\lambda\theta}\theta^{-n\lambda/2} d\theta = \infty$$

This means that $T_{\lambda,\gamma}$ is admissible if $\gamma = 0$ and $\lambda = 2/n$, or if $\gamma > 0$ and $\lambda \geq 2/n$.
In particular, $2Y/(n+2)$ is admissible for estimating $E(T) = 2E(Y)/n = 2\sigma^2$, under the squared error loss.

Example 4.20 (continued)

It is easy to see that $Y/(n+2)$ is then an admissible estimator of σ^2 under the squared error loss and the loss $(a-\sigma^2)^2/\sigma^4$.
Hence $Y/(n+2)$ is minimax under the loss $(a-\sigma^2)^2/\sigma^4$.
Note that we cannot apply Corollary 4.3 directly since $\theta_+ = 0$.

Example 4.21

Let $X_1,...,X_n$ be i.i.d. from the Poisson distribution $P(\theta)$ with an unknown $\theta > 0$.
The joint p.d.f. of X_i's w.r.t. the counting measure is

$$(x_1!\cdots x_n!)^{-1} e^{-n\theta} e^{n\bar{x}\log\theta}$$

For $\eta = n\log\theta$, the conditions of Corollary 4.3 are satisfied with $T(X) = \bar{X}$.
Since $E(T) = \theta$ and $\mathrm{Var}(T) = \theta/n$, by Corollary 4.3, $\bar{X}$ is the unique minimax estimator of θ under the loss function $(a-\theta)^2/\theta$.

Exercise 37 (#4.83)

Let X be an observation from the distribution with Lebesgue density $\frac{1}{2}c(\theta)e^{\theta x-|x|}$, $|\theta|<1$.
(i) Show that $c(\theta)=1-\theta^2$.
(ii) Show that if $0\le\alpha\le\frac{1}{2}$, then $\alpha X+\beta$ is admissible for estimating $E(X)$ under the squared error loss.

Solution

(i) Note that

$$\begin{aligned}
\frac{1}{c(\theta)} &= \frac{1}{2}\int_{-\infty}^{\infty} e^{\theta x-|x|}dx \\
&= \frac{1}{2}\left(\int_{-\infty}^{0} e^{\theta x+x}dx+\int_{0}^{\infty} e^{\theta x-x}dx\right) \\
&= \frac{1}{2}\left(\int_{0}^{\infty} e^{-(1+\theta)x}dx+\int_{0}^{\infty} e^{-(1-\theta)x}dx\right) \\
&= \frac{1}{2}\left(\frac{1}{1+\theta}+\frac{1}{1-\theta}\right)=\frac{1}{1-\theta^2}.
\end{aligned}$$

Solution (continued)

(ii) Consider first $\alpha > 0$. Let $\alpha = (1+\lambda)^{-1}$ and $\beta = \gamma\lambda/(1+\lambda)$.

$$\int_{-1}^{0} \frac{e^{-\gamma\lambda\theta}}{(1-\theta^2)^\lambda} d\theta = \int_{0}^{1} \frac{e^{-\gamma\lambda\theta}}{(1-\theta^2)^\lambda} d\theta = \infty$$

if and only if $\lambda \geq 1$, i.e., $\alpha \leq \frac{1}{2}$.

Hence, $\alpha X + \beta$ is an admissible estimator of $E(X)$ when $0 < \alpha \leq \frac{1}{2}$.

Consider next $\alpha = 0$.

$$\begin{aligned} E(X) &= \frac{1-\theta^2}{2}\left(\int_{-\infty}^{0} x e^{\theta x + x} dx + \int_{0}^{\infty} x e^{\theta x - x} dx\right) \\ &= \frac{1-\theta^2}{2}\left(-\int_{0}^{\infty} x e^{-(1+\theta)} dx + \int_{0}^{\infty} x e^{-(1-\theta)x} dx\right) \\ &= \frac{1-\theta^2}{2}\left(\frac{1+\theta}{1-\theta} - \frac{1-\theta}{1+\theta}\right) = \frac{2\theta}{1-\theta^2}, \end{aligned}$$

which takes any value in $(-\infty, \infty)$.

Hence, the constant estimator β is an admissible estimator of $E(X)$.