Lecture 21: Lengths and expected lengths of confidence intervals

Length criterion

For confidence intervals of a real-valued θ with the same confidence coefficient, an apparent measure of their performance is the interval length.

Shorter confidence intervals are preferred, since they are more informative.

In most problems, however, shortest-length confidence intervals do not exist.

A common approach is to consider a reasonable class of confidence intervals (with the same confidence coefficient) and then find a confidence interval with the shortest length within the class.

When confidence intervals are constructed by using pivotal quantities or by inverting acceptance regions of tests, choosing a reasonable class of confidence intervals amounts to selecting good pivotal quantities or tests.

Example 7.13

Let $X_1, ..., X_n$ be i.i.d. from the uniform distribution $U(0,\theta)$ with an unknown $\theta > 0$.
A confidence interval for θ of the form $[b^{-1}X_{(n)}, a^{-1}X_{(n)}]$ is derived in Example 7.2, where a and b are constants chosen so that this confidence interval has confidence coefficient $1-\alpha$.
Another confidence interval obtained by applying Proposition 7.1 with $T = X$ is of the form $[b_1^{-1}\tilde{X}, a_1^{-1}\tilde{X}]$, where $\tilde{X} = e(\prod_{i=1}^n X_i)^{1/n}$.

We now argue that when n is large enough, the former has a shorter length than the latter.
Since $\sqrt{n}(\tilde{X}-\theta)/\theta \to_d N(0,1)$,

$$
\begin{aligned}
&P\left(\left(1+\frac{d}{\sqrt{n}}\right)^{-1}\tilde{X} \leq \theta \leq \left(1+\frac{c}{\sqrt{n}}\right)^{-1}\tilde{X}\right) \\
=&P\left(\frac{c}{\sqrt{n}} \leq \frac{\tilde{X}-\theta}{\theta} \leq \frac{d}{\sqrt{n}}\right) \to 1-\alpha
\end{aligned}
$$

for some constants c and d.

This means that $a_1 \approx 1 + c/\sqrt{n}$, $b_1 \approx 1 + d/\sqrt{n}$, and the length of $[b_1^{-1}\tilde{X}, a_1^{-1}\tilde{X}]$ converges to 0 a.s. at the rate $n^{-1/2}$.
On the other hand,

$$\begin{aligned} & P\left(\left(1+\tfrac{d}{n}\right)^{-1} X_{(n)} \leq \theta \leq \left(1+\tfrac{c}{n}\right)^{-1} X_{(n)}\right) \\ =& P\left(\tfrac{c}{n} \leq \tfrac{X_{(n)}-\theta}{\theta} \leq \tfrac{d}{n}\right) \to 1-\alpha \end{aligned}$$

for some constants c and d, since $n(X_{(n)} - \theta)/\theta$ has a known limiting distribution (Example 2.34).
This means that the length of $[b^{-1}X_{(n)}, a^{-1}X_{(n)}]$ converges to 0 a.s. at the rate n^{-1} and, therefore, $[b^{-1}X_{(n)}, a^{-1}X_{(n)}]$ is shorter than $[b_1^{-1}\tilde{X}, a_1^{-1}\tilde{X}]$ for sufficiently large n a.s.

Thus, it is reasonable to consider the class of confidence intervals of the form $[b^{-1}X_{(n)}, a^{-1}X_{(n)}]$ subject to $P(b^{-1}X_{(n)} \leq \theta \leq a^{-1}X_{(n)}) = 1 - \alpha$.

The shortest-length interval within this class can be derived as follows. Note that $X_{(n)}/\theta$ has the Lebesgue p.d.f. $nx^{n-1}I_{(0,1)}(x)$.

Hence

$$1-\alpha = P(b^{-1}X_{(n)} \le \theta \le a^{-1}X_{(n)}) = \int_a^b nx^{n-1}dx = b^n - a^n.$$

This implies that $1 \ge b > a \ge 0$ and $\frac{da}{db} = (\frac{b}{a})^{n-1}$.
Since the length of the interval $[b^{-1}X_{(n)}, a^{-1}X_{(n)}]$ is
$\psi(a,b) = X_{(n)}(a^{-1} - b^{-1})$,

$$\frac{d\psi}{db} = X_{(n)}\left(\frac{1}{b^2} - \frac{1}{a^2}\frac{da}{db}\right) = X_{(n)}\frac{a^{n+1} - b^{n+1}}{b^2 a^{n+1}} < 0.$$

Hence the minimum occurs at $b = 1$ ($a = \alpha^{1/n}$).
This shows that the shortest-length interval is $[X_{(n)}, \alpha^{-1/n}X_{(n)}]$.

Shortest confidence interval

For a large class of problems, the following result can be used to find a shortest confidence interval.

Theorem 7.3

Let θ be a real-valued parameter and $T(X)$ be a real-valued statistic.
(i) Let $U(X)$ be a positive statistic.
Suppose that $(T-\theta)/U$ is a pivotal quantity having a Lebesgue p.d.f. f that is *unimodal* at $x_0 \in \mathscr{R}$ in the sense that $f(x)$ is nondecreasing for $x \le x_0$ and $f(x)$ is nonincreasing for $x \ge x_0$.
Consider the following class of confidence intervals for θ:

$$\mathscr{C} = \left\{ [T-bU, T-aU] : \ a \in \mathscr{R}, b \in \mathscr{R}, \int_a^b f(x)dx = 1-\alpha \right\}.$$

If $[T-b_*U, T-a_*U] \in \mathscr{C}$, $f(a_*) = f(b_*) > 0$, and $a_* \le x_0 \le b_*$, then the interval $[T-b_*U, T-a_*U]$ has the shortest length within $\mathscr{C}$.
(ii) Suppose that $T>0$, $\theta>0$, T/θ is a pivotal quantity having a Lebesgue p.d.f. f, and that $x^2f(x)$ is unimodal at x_0.
Consider the following class of confidence intervals for θ:

$$\mathscr{C} = \left\{ [b^{-1}T, a^{-1}T] : \ a>0, b>0, \int_a^b f(x)dx = 1-\alpha \right\}.$$

If $[b_*^{-1}T, a_*^{-1}T] \in \mathscr{C}$, $a_*^2 f(a_*) = b_*^2 f(b_*) > 0$, and $a_* \le x_0 \le b_*$, then the interval $[b_*^{-1}T, a_*^{-1}T]$ has the shortest length within $\mathscr{C}$.

Proof

We prove (i) only.
The proof of (ii) is left as an exercise.
Note that the length of an interval in $\mathscr{C}$ is $(b-a)U$.
Thus, it suffices to show that if $a < b$ and $b-a < b_* - a_*$, then

$$\int_a^b f(x)dx < 1-\alpha.$$

Assume that $a < b$, $b-a < b_* - a_*$, and $a \le a_*$
(the proof for $a > a_*$ is similar).
If $b \le a_*$, then $a \le b \le a_* \le x_0$ and

$$\int_a^b f(x)dx \le f(a_*)(b-a) < f(a_*)(b_* - a_*) \le \int_{a_*}^{b_*} f(x)dx = 1-\alpha,$$

where the first inequality follows from the unimodality of f, the strict inequality follows from $b-a < b_* - a_*$ and $f(a_*) > 0$, and the last inequality follows from the unimodality of f and the fact that $f(a_*) = f(b_*)$.

If $b > a_*$, then $a \leq a_* < b < b_*$.
By the unimodality of f,

$$\int_a^{a_*} f(x)dx \leq f(a_*)(a_* - a) \qquad \text{and} \qquad \int_b^{b_*} f(x)dx \geq f(b_*)(b_* - b).$$

Then

$$\begin{aligned}
\int_a^b f(x)dx &= \int_{a_*}^{b_*} f(x)dx + \int_a^{a_*} f(x)dx - \int_b^{b_*} f(x)dx \\
&= 1 - \alpha + \int_a^{a_*} f(x)dx - \int_b^{b_*} f(x)dx \\
&\leq 1 - \alpha + f(a_*)(a_* - a) - f(b_*)(b_* - b) \\
&= 1 - \alpha + f(a_*)[(a_* - a) - (b_* - b)] \\
&= 1 - \alpha + f(a_*)[(b - a) - (b_* - a_*)] \\
&< 1 - \alpha.
\end{aligned}$$

This completes the proof.

Example 7.14

Let $X_1, ..., X_n$ be i.i.d. from $N(\mu, \sigma^2)$ with unknown μ and σ^2.
Confidence intervals for $\theta = \mu$ using the pivotal quantity $\sqrt{n}(\bar{X} - \mu)/S$ form the class $\mathscr{C}$ in Theorem 7.3(i) with f being the p.d.f. of the t-distribution t_{n-1}, which is unimodal at $x_0 = 0$.
Hence, we can apply Theorem 7.3(i).
Since f is symmetric about 0, $f(a_*) = f(b_*)$ implies $a_* = -b_*$ (exercise).
Therefore, the equal-tail confidence interval

$$\left[\bar{X} - t_{n-1,\alpha/2} S/\sqrt{n}, \bar{X} + t_{n-1,\alpha/2} S/\sqrt{n}\right]$$

has the shortest length within $\mathscr{C}$.

If $\theta = \mu$ and σ^2 is known, then we can replace S by σ and f by the standard normal p.d.f. (i.e., use the pivotal quantity $\sqrt{n}(\bar{X} - \mu)/\sigma$ instead of $\sqrt{n}(\bar{X} - \mu)/S$).
The resulting confidence interval is

$$\left[\bar{X} - \Phi^{-1}(1-\alpha/2)\sigma/\sqrt{n}, \bar{X} + \Phi^{-1}(1-\alpha/2)\sigma/\sqrt{n}\right],$$

which is the shortest interval of the form $[\bar{X} - b, \bar{X} - a]$ with confidence coefficient $1 - \alpha$.

The length difference of the two confidence intervals is a random variable so that we cannot tell which one is better in general.
But the expected length of the second interval is always shorter than that of the first interval (exercise).
This again shows the importance of picking the right pivotal quantity.

Consider next confidence intervals for $\theta = \sigma^2$ using the pivotal quantity $(n-1)S^2/\sigma^2$, which form the class $\mathscr{C}$ in Theorem 7.3(ii) with f being the p.d.f. of the chi-square distribution χ^2_{n-1}.
Note that $x^2 f(x)$ is unimodal, but not symmetric.
By Theorem 7.3(ii), the shortest-length interval within $\mathscr{C}$ is

$$[\, b_*^{-1}(n-1)S^2, a_*^{-1}(n-1)S^2 \,],$$

where a_* and b_* are solutions of $a_*^2 f(a_*) = b_*^2 f(b_*)$ and $\int_{a_*}^{b_*} f(x)dx = 1-\alpha$.
Numerical values of a_* and b_* can be obtained (Tate and Klett, 1959).
Note that this interval is not equal-tail.

Suppose that we need a confidence interval for $\theta = \sigma$ when μ is unknown.

Consider the class of confidence intervals

$$\left[b^{-1/2}\sqrt{n-1}\,S, a^{-1/2}\sqrt{n-1}\,S \right]$$

with $\int_a^b f(x)dx = 1-\alpha$ and f being the p.d.f. of χ^2_{n-1}.
The shortest-length interval, however, is not the one with the endpoints equal to the square roots of the endpoints of the interval

$$[\, b_*^{-1}(n-1)S^2, a_*^{-1}(n-1)S^2 \,]$$

for σ^2 (Exercise 36(c)).

Remarks

- Note that Theorem 7.3(ii) cannot be applied to obtain the result in Example 7.13 unless $n=1$, since the p.d.f. of $X_{(n)}/\theta$ is strictly increasing when $n>1$; see Exercise 38.
- The result in Theorem 7.3 can be applied to justify the idea of HPD credible sets in Bayesian analysis (Exercise 40).
- If a confidence interval has the shortest length within a class of confidence intervals, then its expected length is also the shortest within the same class, provided that its expected length is finite.

Expected length

In a problem where a shortest-length confidence interval does not exist, we may have to use the expected length as the criterion in comparing confidence intervals.

For instance, the expected length of $\left[\bar{X}-\Phi^{-1}(1-\alpha/2)\sigma/\sqrt{n}, \bar{X}+\Phi^{-1}(1-\alpha/2)\sigma/\sqrt{n}\right]$ is always shorter than that of $\left[\bar{X}-t_{n-1,\alpha/2}S/\sqrt{n}, \bar{X}+t_{n-1,\alpha/2}S/\sqrt{n}\right]$, whereas the probability that $\left[\bar{X}-t_{n-1,\alpha/2}S/\sqrt{n}, \bar{X}+t_{n-1,\alpha/2}S/\sqrt{n}\right]$ is shorter than $\left[\bar{X}-\Phi^{-1}(1-\alpha/2)\sigma/\sqrt{n}, \bar{X}+\Phi^{-1}(1-\alpha/2)\sigma/\sqrt{n}\right]$ is positive for any fixed n.

Another example is the interval $[X_{(n)}, \alpha^{-1/n}X_{(n)}]$ in Example 7.13. Although we are not able to say that this interval has the shortest length among all confidence intervals for θ with confidence coefficient $1-\alpha$, we can show that it has the shortest expected length, using the results in Theorems 7.4 and 7.6 (§7.2.2).

The following result is more general than the one about expected lengths of confidence intervals.

Volumn

The volume of a confidence set $C(X)$ for $\theta \in \mathscr{R}^k$ when $X = x$ is defined to be

$$\text{vol}(C(x)) = \int_{C(x)} d\theta',$$

which is the Lebesgue measure of the set $C(x)$ and may be infinite.
In particular, if θ is real-valued and $C(X) = [\underline{\theta}(X), \overline{\theta}(X)]$ is a confidence interval, then $\text{vol}(C(x))$ is simply the length of $C(x)$.
The next result reveals a relationship between the expected volume (length) and the probability of covering a false value of a confidence set (interval).

Theorem 7.6 (Pratt's theorem)

Let X be a sample from P and $C(X)$ be a confidence set for $\theta \in \mathscr{R}^k$.
Suppose that $\text{vol}(C(x)) = \int_{C(x)} d\theta'$ is finite a.s. P.
Then the expected volume of $C(X)$ is

$$E[\text{vol}(C(X))] = \int_{\theta \neq \theta'} P\big(\theta' \in C(X)\big) d\theta'.$$

Proof

By Fubini's theorem,

$$\begin{aligned} E[\mathrm{vol}(C(X))] &= \int \mathrm{vol}(C(X))dP \\ &= \int \left[\int_{C(x)} d\theta'\right] dP(x) \\ &= \int\int_{\theta'\in C(x)} d\theta' dP(x) \\ &= \int \left[\int_{\theta'\in C(x)} dP(x)\right] d\theta' \\ &= \int P\big(\theta' \in C(X)\big) d\theta' \\ &= \int_{\theta\neq\theta'} P\big(\theta' \in C(X)\big) d\theta'. \end{aligned}$$

This proves the result.

Discussions

It follows from Theorem 7.6 that if $C(X)$ is UMA (or UMAU) with confidence coefficient $1-\alpha$, then it has the smallest expected volume among all confidence sets (or all unbiased confidence sets) with confidence level $1-\alpha$.
For example, the confidence interval in Example 7.14 (when σ^2 is known) or $[X_{(n)}, \alpha^{-1/n}X_{(n)}]$ in Example 7.13 has the shortest expected length among all confidence intervals with confidence level $1-\alpha$.

Example

Let X_{ij}, $i = 1, ..., n_j$, $j = 1, 2$, be independent random variables from the exponential distributions on $(0, \infty)$ with scale parameters θ_j.
We want to derive the UMAU confidence interval for θ_2/θ_1 with confidence coefficient $1-\alpha$.
First, we need to find a UMPU test of size α for testing $H_0 : \theta_2 = \lambda\theta_1$ versus $H_1 : \theta_2 \neq \lambda\theta_1$, where $\lambda > 0$ is a known constant.
The joint density of $X_1 = \sum_{i=1}^{n_1} X_{i1}$ and $X_2 = \sum_{i=1}^{n_2} X_{i2}$ is

$$\frac{X_1^{n_1-1}X_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\theta_1^{n_1}\theta_2^{n_2}} \exp\left\{-\frac{X_1}{\theta_1} - \frac{X_2}{\theta_2}\right\},$$

which can be written as

$$\frac{X_1^{n_1-1}X_2^{n_2-1}}{\Gamma(n_1)\Gamma(n_2)\theta_1^{n_1}\theta_2^{n_2}}\exp\left\{-X_1\left(\frac{1}{\theta_1}-\frac{\lambda}{\theta_2}\right)-(\lambda X_1+X_2)\frac{1}{\theta_2}\right\}.$$

Hence, by Theorem 6.4, a UMPU test of size α rejects H_0 when $X_1 < c_1(U)$ or $X_1 > c_2(U)$, where $U = \lambda X_1 + X_2$.

Note that X_1/X_2 is independent of U under H_0.

Hence, by Lemma 6.7, the UMPU test is equivalent to the test that rejects H_0 when $X_1/X_2 < d_1$ or $X_1/X_2 > d_2$, which is equivalent to the test that rejects H_0 when $W < b_1$ or $W > b_2$, where $W = \frac{Y/\lambda}{1+Y/\lambda}$, $Y = X_2/X_1$, and b_1 and b_2 satisfy $P(b_1 < W < b_2) = 1-\alpha$ (for size α) and $E[WI_{(b_1,b_2)}(W)] = (1-\alpha)E(W)$ (for unbiasedness) under H_0.

When $\theta_2 = \lambda\theta_1$, W has the beta distribution $B(n_1, n_2)$, and hence b_1 and b_2 can be calculated.

Hence, the acceptance region of the UMPU test is

$$A(\lambda) = \{W : b_1 \le W \le b_2\} = \left\{Y : \frac{b_1}{1-b_1} \le \frac{Y}{\lambda} \le \frac{b_2}{1-b_2}\right\}.$$

Inverting $A(\lambda)$ leads to

$$C(X) = \{\lambda : \lambda \in A(\lambda)\} = \left[\frac{(1-b_2)Y}{b_2}, \frac{(1-b_1)Y}{b_1}\right],$$

which is a UMAU confidence interval for $\lambda = \theta_2/\theta_1$ with confidence coefficient $1-\alpha$.

By Theorem 7.6, interval $C(X)$ has the shortest expected length among all unbiased confidence intervals with confidence coefficient $1-\alpha$.

The expected length of $C(X)$ is

$$\begin{aligned}\left(\frac{1-b_1}{b_1} - \frac{1-b_2}{b_2}\right) E(Y) &= \left(\frac{1}{b_1} - \frac{1}{b_2}\right) E\left(\frac{X_2}{X_1}\right) \\ &= \left(\frac{1}{b_1} - \frac{1}{b_2}\right) E(X_2) E\left(\frac{1}{X_1}\right) \\ &= \left(\frac{1}{b_1} - \frac{1}{b_2}\right) \frac{n_2\theta_2}{(n_1-1)\theta_1}\end{aligned}$$

provided that $n_1 > 1$.