Power and Sample Size Determination

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October 28, 2010

Experimental Design

- To this point in the semester, we have largely focused on methods to analyze *the data that we have* without any regard to *from where the data comes.*
- *Design of Experiments* is the area of statistics that examines plans on how to gather data to achieve good (or optimal) inference.
- Here, we will focus on the question of sample size:
  - how large does a sample need to be so that a confidence interval will be no wider than a given size?
  - how large does a sample need to be so that a hypothesis test will have a low p-value if a certain alternative hypothesis is true?
- Sample size determination is just one aspect of good design of experiments: *we will encounter additional aspects next lecture*.

Proportions

- Recall methods for inference about proportions: confidence intervals

Confidence Interval for p

A P% confidence interval for p is

$$p' - z^* \sqrt{\frac{p'(1-p')}{n'}} < p < p' + z^* \sqrt{\frac{p'(1-p')}{n'}}$$

where $n' = n + 4$ and $p' = \frac{X+2}{n+4} = \frac{X+2}{n'}$ and z^* is the critical number from a standard normal distribution where the area between $-z^*$ and z^* is $P/100$. (For 95%, $z^* = 1.96$.)

Proportions

- . . . and hypothesis tests.

The Binomial Test

If $X \sim \mathrm{Binomial}(n, p)$ with null hypothesis $p = p_0$ and we observe $X = x$, the p-value is the probability that a new random variable $Y \sim \mathrm{Binomial}(n, p_0)$ would be at least as extreme (either $\mathrm{P}(Y \leq x)$ or $\mathrm{P}(Y \geq x)$ or $\mathrm{P}(|Y - np_0| \geq |x - np_0|)$ depending on the alternative hypothesis chosen.)

Sample size for proportions

- On November 2, Wisconsin will hold an election with races for the U.S. Senate and the Governor.
- A news organization plans to take a final poll over the weekend.
- One population proportion of interest is the proportion of voters who will cast a ballot for incumbent senator Russ Feingold.
- Assuming that they can take a random sample of likely voters:

  How large of a sample is needed for a 95% confidence interval to have a margin of error of no more than 4%?

Calculation

- Notice that the margin of error depends on both n and p', but we do not know p'.

$$1.96\sqrt{\frac{p'(1-p')}{n+4}}$$

- However, the expression $p'(1-p')$ is maximized at 0.5; if the value of p' from the sample turns out to be different, the margin of error will just be a bit smaller, which is even better.
- So, it is conservative (in a statistical, not political sense) to set $p' = 0.5$ and then solve this inequality for n.

$$1.96\sqrt{\frac{(0.5)(0.5)}{n+4}} < 0.04$$

- Show on the board why $n > \left(\frac{(1.96)(0.5)}{0.04}\right)^2 - 4 \doteq 621$.

General Formula

Sample size for proportions

$$n > \left(\frac{(1.96)(0.5)}{M}\right)^2 - 4$$

where M is the desired margin of error.

Significance Level and Power

Definition

The *significance level of a test* is the probability of rejecting the null hypothesis when it is true and is denoted α.

Definition

The *power* of a hypothesis test for a specified alternative hypothesis is $1 - \beta$, which is the probability of rejecting a specific true alternative hypothesis.

- Note that as there are many possible alternative hypotheses, for a single α there are many values of β.
- It is helpful to plot the probability of rejecting the null hypothesis against the parameter.

Example

- Consider a population with proportion p.
- Let X be the number of successes in a random sample of size 100
- with model $X \sim \mathrm{Binomial}(100, p)$.
- Consider the hypotheses $H_0 \colon p = 0.3$ versus $H_A \colon p < 0.3$ and the decision to reject if $X \leq 22$.
  1. Find α
  2. Find β if $p = 0.2$.
  3. Plot the probability of rejecting the null hypothesis versus p.

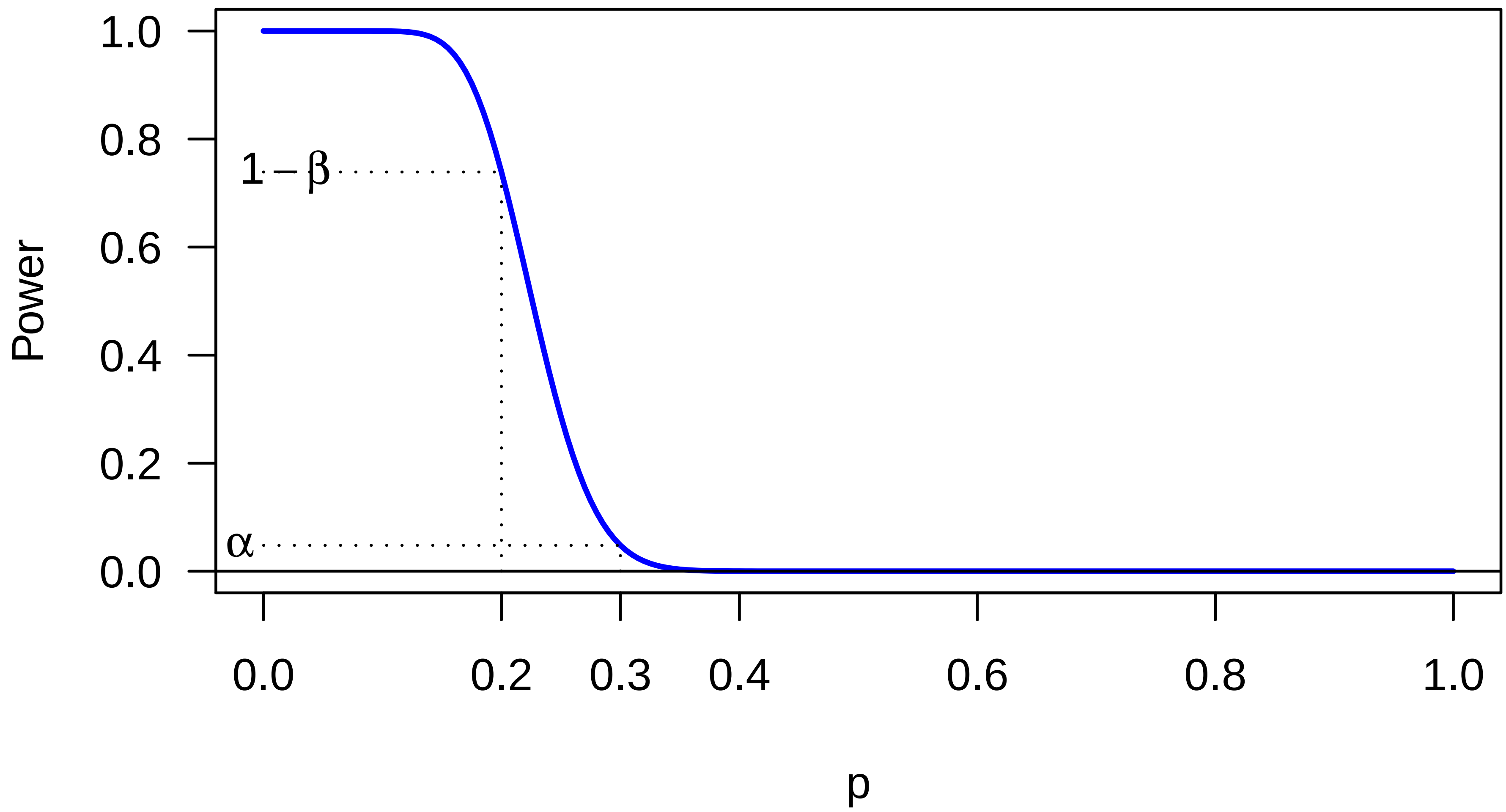
Solution

$$\begin{aligned}
\alpha &= \mathrm{P}(X \le 22 \mid p = 0.3) \\
&= \sum_{k=0}^{22} \binom{100}{k} (0.3)^k (0.7)^{100-k} \\
&\doteq 0.0479
\end{aligned}$$

$$\begin{aligned}
1 - \beta(p) &= \mathrm{P}(X \le 22 \mid p) \\
&= \sum_{k=0}^{22} \binom{100}{k} p^k (1-p)^{100-k}
\end{aligned}$$

$$1 - \beta(0.2) = 0.7389$$

Graph of Power



Sample Size

- Suppose that we wanted a sample size large enough so that we could pick a rejection rule where α was less than 0.05 and the power when $p = 0.2$ was greater than 0.9.
- How large would n need to be?
- Simplify by letting n be a multiple of 25.
- We see in the previous example that 100 is not large enough: if the critical rejection value were more than 22, then $\alpha > 0.05$ and the power is 0.7389 which is less than 0.9.
- There is no simple formula, but we can use R to find the answer by trial and error.

Calculation

- Here we consider $n = 150$ and find the best rejection k such that $\mathrm{P}(X \leq k \mid p = 0.3) < 0.05$ when $n = 150$. The answer is $k = 35$.

```
> cbind(30:40, pbinom(30:40, 150, 0.3))
```

```
      [,1]        [,2]
 [1,]   30 0.003819432
 [2,]   31 0.006657905
 [3,]   32 0.011181720
 [4,]   33 0.018114321
 [5,]   34 0.028338450
 [6,]   35 0.042860887
 [7,]   36 0.062742795
 [8,]   37 0.088996125
 [9,]   38 0.122454316
[10,]   39 0.163633628
[11,]   40 0.212607595
```

Calculation

- Here is a trickier way to do it.

```
> k = max(which(pbinom(1:150, 150, 0.3) < 0.05))
> k
```

```
[1] 35
```

- Find α and $1 - \beta$ for this value of k.

```
> alpha = pbinom(k, 150, 0.3)
> power = pbinom(k, 150, 0.2)
> c(alpha, power)
```

```
[1] 0.04286089 0.86831829
```

- The power is not quite high enough. Try $n = 175$.

Calculation

- This code finds the largest value k such that $\mathrm{P}(X \leq k \mid p = 0.3) < 0.05$ when $n = 175$.

```
> k = max(which(pbinom(1:175, 175, 0.3) < 0.05))
> k

[1] 42
```

- Find α and $1 - \beta$ for this value of k.

```
> alpha = pbinom(k, 175, 0.3)
> power = pbinom(k, 175, 0.2)
> c(alpha, power)

[1] 0.04733449 0.91935706
```

- We see here that $n = 175$ is large enough for a test with significance level less than 0.05 when the null $p = 0.3$ is true will have power at least 0.9 to reject H_0 when $p = 0.2$.

Normal Populations

- The previous problems were for the binomial distribution and proportions, which is tricky because of the discreteness and necessary sums of binomial probability calculations.
- Answering similar problems for normal populations is easier.
- However, we need to provide a guess for σ.

Example

- We want to know the mean percentage of butterfat in milk produced by area farms.
- We can sample multiple loads of milk.
- Previous records indicate that the standard deviation among loads is 0.15 percent.
- How many loads should we sample if we desire the margin of error of a 99% confidence interval for μ, the mean butterfat percentage, to be no more than 0.03?

Confidence Intervals

- We will use the t distribution, not the standard normal for the actual confidence interval, but it is easiest to use the normal distribution for planning the design.
- If the necessary sample size is even moderately large, the differences between z^* and t^* is tiny.
- Recall the confidence interval for μ.

Confidence Interval for μ

A P% confidence interval for μ has the form

$$\bar{Y} - t^* \frac{s}{\sqrt{n}} < \mu < \bar{Y} + t^* \frac{s}{\sqrt{n}}$$

where t^* is the critical value such that the area between $-t^*$ and t^* under a t-density with $n-1$ degrees of freedom is $P/100$, where n is the sample size.

Calculation

- For a 99% confidence interval, we find $z^* = 2.58$.
- We need n so that
$$2.58 \times \frac{0.15}{\sqrt{n}} < 0.03$$
- Work on the board to show $n > \left(\frac{(2.58)(0.15)}{0.03}\right)^2 \doteq 167$.

General Formula

Sample size for a single mean

$$n > \left(\frac{(z^*)(\sigma)}{M}\right)^2$$

where M is the desired margin of error.

Example

- Graph the power for a sample size of $n = 25$ for $\alpha = 0.05$ for $H_0\colon \mu = 3.35$ versus $H_A\colon \mu \neq 3.35$.
- Note that the p-value is less than 0.05 approximately when

$$\left| \frac{\bar{X} - 3.35}{0.15/\sqrt{25}} \right| > 1.96$$

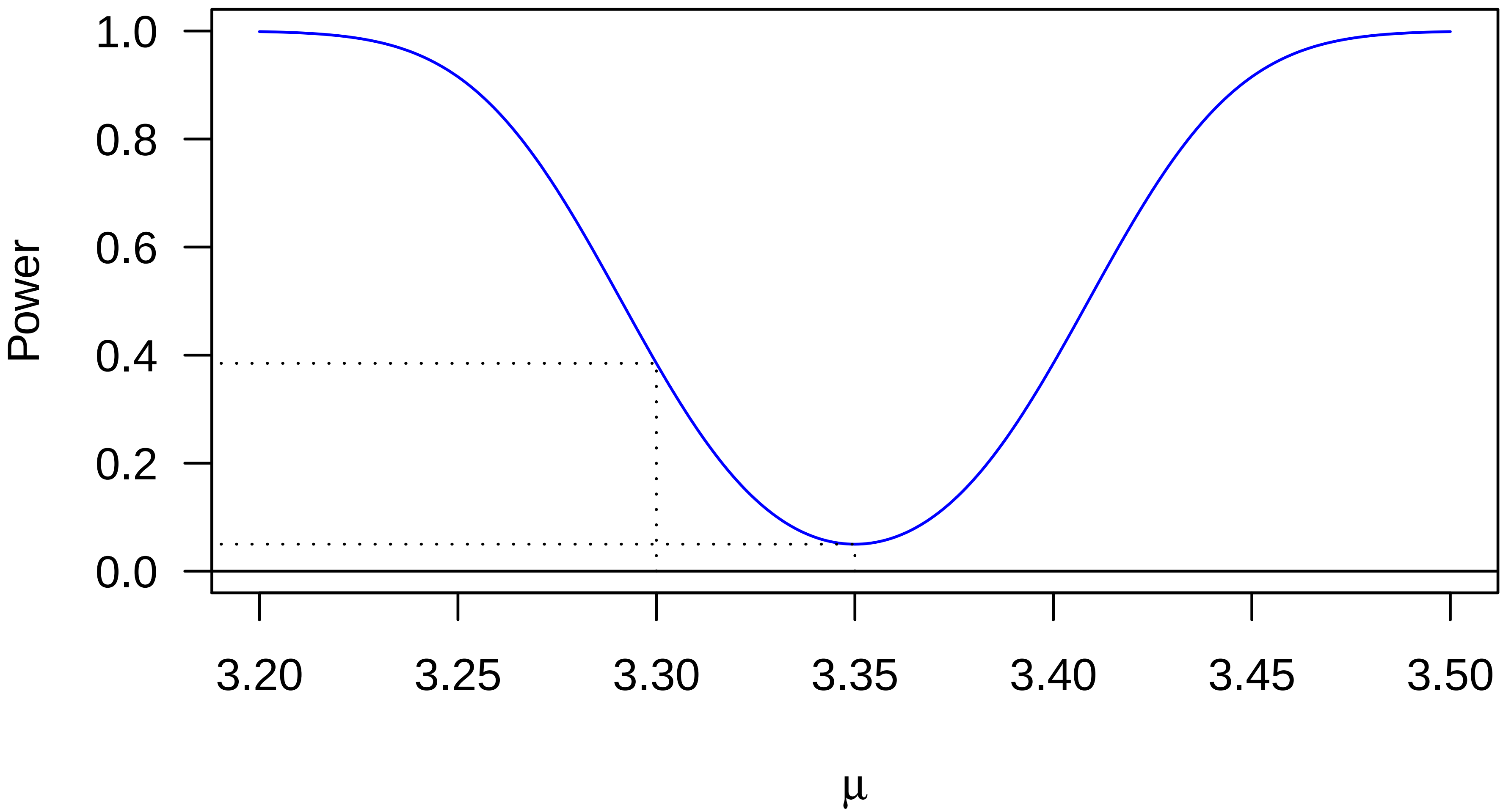
- The rejection region is

$$\bar{X} < 3.35 - 1.96 \frac{0.15}{\sqrt{25}} \doteq 3.291$$

or

$$\bar{X} > 3.35 + 1.96 \frac{0.15}{\sqrt{25}} \doteq 3.409$$

Power: $n = 25$, $H_0 : \mu = 3.35$



Problem

- Long-run average percent butterfat in milk at a farm is 3.35 and the standard deviation is 0.15, with measurements taken by the load.
- How large should a sample size be to have an 80% chance of detecting a change to $\mu = 3.30$ at a significance level of $\alpha = 0.05$ with a two-sided test?
- We can explore this informally by graphing power curves for different n until we find a solution.
- Homework will guide you to an algebraic solution.

What you should know

You should know:

- what the definitions of power and significance level are;
- how to find sample sizes for desired sizes of confidence intervals;
- how to find sample sizes with desired power for specific alternative hypotheses;
- how to examine and interpret a power curve;
- how power curves change as n changes.