Discussion 9

Review – Power and Sample Size

General formula of the sample size such that a $P\%$ confidence interval constructed with margin of error smaller than M.
Sample size for proportion:

$$n > (\frac{z^* 0.5}{M})^2 - 4$$

where M is the desired margin of error, z* is the critical number from a standard normal distribution with the area between $-z^*$ and z^* as $P/100$. (For 95%, $z^* = 1.96$. For 90%, $z^* = 1.64$)

Sample size for a single mean:

$$n > (\frac{z^* \sigma}{M})^2$$

where M is the desired margin of error.

The *significance level of a test* is the probability of rejecting the null hypothesis when it is true and is denoted α.

The *power* of a hypothesis test for a specified alternative is $1 - \beta$, which is the probability of make a rejection when alternative is true.

Practice Problems

Problem 1

We want to know the mean percentage of butterfat in milk produced by a farm by sampling multiple loads of milk. Previous records indicate the average percent butterfat in milk is 3.35 and the standard deviation among loads is 0.15. Now we hope to detect a change of the percent butterfat in milk to 3.40 with a power 0.8.

(a) How many loads do we need to sample so that the margin of error for a 90% confidence interval of the mean percent butterfat is no more than 0.06?

(b) Suppose 100 loads of milk are sampled. Find the 0.975 and 0.025 quantile of the sampling distribution of $\bar{X}$ assuming $\mu = 3.35$. Find the rejection region at the significance level $\alpha = 0.05$. What is the power of the test for detecting a change of the mean to 3.40?

(c) Suppose the sample size is n. Let a denote 0.975 quantile of the sampling distribution of $\bar{X}$ assuming $\mu = 3.35$. Express a as a function of sample size n. (Note that it is just the boundary of the upper rejection rejection when $\alpha = 0.05$ with a sample size n.)

(d) Following part (c), if the power is 0.8 when $\mu = 3.40$, then a will also be approximately the 0.2 quantile for the $N(3.40, (0.15^2/n))$ distribution. Express a as a function of n.

(e) Set these expressions for a equal to one another and solve n. How large does the sample size need to be to meet the criteria.

Problem2

This question concerns the comparison of two different rejection rules for evaluating a hypothesis. The data consist of measurements of the breaking strength of wooden boards. For each board, an increasing force is applied until the board breaks. The measurement is the amount of force required to break the board.

A random sample of 100 observations of breaking force is available. Assume that these observations follow a $N(\mu, 800)$ distribution. Of interest is the null hypothesis $H_0 : \mu = 56$ versus the alternative $H_0 : \mu > 56$. Two different rejection rules are being considered; each has roughly the same value of α.

Rule A: Reject H_0 if $\bar{X} > 60$.

Rule B: For each observation, X_i, declare the observation to be "defective" if $X_i > 70$. Reject H_0 if the number of defective observations is ≥ 36.

If, in fact, $\mu = 62$, find the power for each rule and determine which is more powerful.

Problem 3

A scientist wishes to compare two drugs in terms of the glucose levels in the anterior chambers of the eyes of dogs. There will be n dogs in the study. For each dog, one of the drugs will be randomly assigned to the right eye and the other to the left. The difference in glucose levels are known to be approximately normally distributed with a variance of 1.8 $(mg/dLi)^2$. The null hypothesis is that the two drugs have the same effect versus the one-sided alternative that drug B results in a higher glucose level. The hypothesis will be rejected if the observed difference between the sample mean for drug B and that for Drug A is 0.3 mg/dLi or greater. How large does n need to be so that there is a power of 90% when the true difference in glucose level is 0.5 mg/dLi?

Solution

Q1

(a)$z^* = 1.64$, $M = 0.06$, $\sigma = 0.15$. By $n > (\frac{z^*\sigma}{M})^2$,

$$n > (\frac{1.64 * 0.15}{0.06})^2$$

which is $n > 16.81$. Thus the number of loads sampled is at least 17.

(b) The sampling distribution of $\bar{X}$ is a normal distribution with mean 3.35 and std $0.15/\sqrt{100} = 0.015$. Thus the 0.975 and 0.025 quantile are respectively

$$3.35 + 1.96 * 0.015 = 3.379$$

and

$$3.35 + 1.96 * 0.015 = 3.3206$$

Thus the rejection region at a significance level $\alpha = 0.05$ is $(-\inf, 3.3206)$ or $(3.379, +\inf)$.

The power under the specified alternative with mean $\mu = 3.40$ is

$$P(\bar{X} > 3.379|\mu = 3.40) = P(\frac{\bar{X} - 3.40}{0.015} > \frac{3.379 - 3.40}{0.015}) = P(Z > -1.4) = 0.91$$

(c) $a = 3.35 + 1.96 * \frac{0.15}{\sqrt{n}}$

(d) The 0.2 quantile of a standard normal is -0.84.
Meanwhile, a should also satisfy the equation $a = 3.40 - 0.84 * \frac{0.15}{\sqrt{n}}$.

(e) Set $3.35 + 1.96 * \frac{0.15}{\sqrt{n}} = 3.40 - 0.84 * \frac{0.15}{\sqrt{n}}$. We derive that $\sqrt{n} = \frac{0.15*(1.96+0.84)}{0.05} = 8.4$.
Therefore $n > 70.56$. The sample size is at least 71 to detect a change of mean to 3.40 with power 0.8.

Q2

For Rule A: $\bar{X} \sim N(62, 800/100)$, and the power is

$$\begin{aligned}
&P(\bar{X} > 60) \\
&= P\left(Z > \frac{60 - 62}{\sqrt{8}}\right) \\
&= P(Z > -0.71) \\
&= 0.7611.
\end{aligned}$$

For Rule B, we have

$$\begin{aligned}
&P(X_i > 70) \\
&= P\left(Z > \frac{70 - 62}{\sqrt{800}}\right) \\
&= P(Z > 0.28) \\
&= 0.3817.
\end{aligned}$$

Denote the number of the defective observations as W. Then $W \sim B(100, 0.3817) \approx N(38.97, 23.78)$, and the power is

$$
\begin{aligned}
& P(W > 36) \\
&= P\left(Z > \frac{36 - 38.97}{\sqrt{23.78}}\right) \\
&= P(Z > -0.61) \\
&= 0.7291.
\end{aligned}
$$

Hence, Rule A is more powerful.

Q3

The random variable of interest in D, the difference in glucose levels between B and A. The stated condition requires the distribution of D under the given alternative. In this case, $D \sim N(.5, 1.8)$. The given condition is that $P(\bar{D} \geq 0.3|\mu_D = 0.5) = 0.9$. We know $P(Z \geq -1.28) = 0.9$. Thus, $-1.28 = (.3 - .5)/\sqrt{1.8/n}$. Solving results in $\sqrt{n} = 8.59$ and, rounding up, $n = 74$.