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ON THE DISTRIBUTION OF SOME STATISTICS USEFUL  
IN THE ANALYSIS OF JOINTLY STATIONARY TIME  
SERIES

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# ON THE DISTRIBUTION OF SOME STATISTICS USEFUL IN THE ANALYSIS OF JOINTLY STATIONARY TIME SERIES\*

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## 1. Introduction and Summary

Let  $\{X(t), t = \dots -1, 0, 1, \dots\}$  be a  $P$  dimensional zero mean stationary Gaussian time series,

$$X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_P(t) \end{pmatrix}$$

we let  $R(\tau) = E X(t) X'(t+\tau)$ , where  $R(\tau) = \{R_{ij}(\tau), i, j = 1, 2, \dots, P\}$ , and  $F(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} e^{-i\omega\tau} R(\tau)$ . It is assumed that  $\sum_{i,j=1}^P \sum_{\tau=-\infty}^{\infty} |\tau| |R_{ij}(\tau)| < \infty$ , and hence  $F(\omega)$  exists and the elements possess bounded derivatives. It is further assumed that  $F(\omega)$  is strictly positive definite, all  $\omega$ . Knowledge of  $F(\omega)$  serves to specify the process.

$F(\omega)$ , and  $S$ , the covariance matrix of  $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_P \end{pmatrix}$ , a Normal  $(0, S)$  random vector are known to enjoy many analogous properties. (See [7].) To cite two examples, the hypothesis that  $X_i(s)$  is independent of  $X_j(t)$  for  $i \neq j = 1, 2, \dots, P$ , any  $s, t$ , is equivalent to the hypothesis that  $F(\omega)$  is diagonal, all  $\omega$ , while the hypothesis that  $x_i$  is independent of  $x_j$ , for  $i \neq j = 1, 2, \dots, P$  is equivalent to the hypothesis that  $S$  is diagonal. The conditional expectation of  $x_1$ , given  $x_2, \dots, x_P$  is

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$$E(x_1 | x_2, \dots, x_p) = S_{12} S_{22}^{-1} \begin{pmatrix} x_2 \\ \vdots \\ x_p \end{pmatrix}, \quad S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

The corresponding regression problem for stationary Gaussian time series goes as follows. If

$$\begin{aligned} E\{X_1(t) \mid X_2(s), \dots, X_p(s), s = \dots, -1, 0, 1, \dots\} \\ = \sum_{j=2}^P \sum_{s=-\infty}^{\infty} b_j(t-s) X_j(s) \end{aligned}$$

then  $B(\omega)$ , defined by

$$B(\omega) = (B_2(\omega), \dots, B_p(\omega)), \quad B_j(\omega) = \sum_{s=-\infty}^{\infty} b_j(s) e^{i\omega s}$$

satisfies

$$B(\omega) = F_{12}(\omega) F_{22}^{-1}(\omega), \quad F(\omega) = \begin{pmatrix} f_{11}(\omega) & f_{12}(\omega) \\ f_{21}(\omega) & f_{22}(\omega) \end{pmatrix}$$

It is interesting to ask how well these and similar analogies carry over to sampling theory and hypothesis testing. Goodman [3] gave a heuristic argument to support the conclusion that  $\hat{F}(\omega_k)$ , a suitably formed estimate of the spectral density matrix  $F(\omega_k)$  has the complex Wishart distribution. The question is met here by the following results. Firstly if  $\hat{F}(\omega_\ell)$ ,  $\ell = 1, 2, \dots, M$  are estimates of the spectral density matrix, each consisting of averages of  $(2n+1)$  periodograms based on a record of length  $T$ , with the  $\omega_\ell$  equally spaced and  $(2n+1)M \leq \frac{T}{2}$ , then it is possible to construct, on the same sample space as  $X(t)$ ,  $M$  independent complex Wishart matrices  $\hat{\tilde{F}}(\omega_\ell)$ ,  $\ell = 1, 2, \dots, M$  such that  $\{\hat{F}(\omega_\ell), \ell = 1, 2, \dots, M\}$  converge simultaneously in mean square to  $\{\hat{\tilde{F}}(\omega_\ell), \ell = 1, 2, \dots, M\}$ , as  $n, M$  get large. Secondly, it is legitimate to use the natural analogies from multivariate analysis to test hypotheses about time series. One example is presented, as follows. The likelihood ratio test

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

where  $\Sigma_{ij} = E X_i X_j'$ ,  $\{\hat{F}(\omega_\ell)\}$ , the sample spectral density matrices described above based on a record of length  $T$ , are each of the form  $\hat{F}(\omega_\ell) = \frac{1}{T} X Q X'$  where  $Q$  is a  $T \times T$  circulant matrix with largest eigenvalue  $= \frac{T}{(2n+1)} \leq \frac{M}{2} < T$ . We define circulant matrices  $\tilde{\Sigma}_{ij}$  which approximate  $\Sigma_{ij}$ , and a random matrix  $\tilde{X}$  on the sample space of  $X$ ,

$$\tilde{X} = \begin{pmatrix} \tilde{X}_1 \\ \vdots \\ \tilde{X}_2 \end{pmatrix} = \begin{pmatrix} \tilde{X}_1(1), & \dots & \tilde{X}_1(T) \\ \tilde{X}_2(1), & & \tilde{X}_2(T) \end{pmatrix}$$

with  $E \tilde{X}_i \tilde{X}_j' = \tilde{\Sigma}_{ij}$ . The  $2T$  eigenvalues of the block circulant matrix

$$\tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} \\ \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} \end{pmatrix}$$

will be the  $2T$  eigenvalues of the  $T$  matrices  $\{F(\frac{2\pi j}{T}), j = 1, 2, \dots, T\}$ . The distribution of random matrices of the form  $\frac{1}{T} \tilde{X} Q \tilde{X}'$  where  $Q$  is any circulant matrix is relatively simple to investigate due to the fact that all circulant matrices commute, and their eigenvalues may be exhibited as simple functions of the elements. Circulant quadratic forms in random vectors with circulant covariance matrices are well known in the literature, (See [1] and references cited there). Let  $\hat{F}_{X,Q} = \frac{1}{T} X Q X'$  and  $\hat{F}_{\tilde{X},Q} = \frac{1}{T} \tilde{X} Q \tilde{X}'$  where  $Q$  is now any  $T \times T$  (real or complex) quadratic form with largest absolute eigenvalue  $\leq q$ . The main Theorem allows the replacement of  $X$  by  $\tilde{X}$  in the analysis, and is, that under the assumptions on  $F(\omega)$  and  $R(\tau)$ , for any  $T$ ,

$$(1.1) \quad E \text{Tr} (\hat{F}_{X,Q} - \hat{F}_{\tilde{X},Q}) (\hat{F}_{X,Q} - \hat{F}_{\tilde{X},Q})^{*'} \leq c \frac{q^2}{T^2}$$

where  $c$  is a constant depending only on  $F(\omega)$  and  $R(\tau)$ . A lemma, essentially allowing the replacement of  $F(\omega)$  by a suitably chosen step-function, together

Since

$$\begin{aligned} \frac{1}{T} \sum_{v=1}^T e^{\frac{2\pi i v}{T} (r-s-\tau)} &= 1 & \tau = (r-s) + \ell T, \quad \ell = 0, \pm 1, \pm 2, \dots, \\ &= 0 & \text{otherwise,} \end{aligned}$$

we have

$$\begin{aligned} [2\pi W D_K W^*]_{r,s} &= q_{r-s}, & r-s \geq 0, \\ &= q_{T-|(r-s)|}, & r-s < 0. \end{aligned}$$

Remarks.  $Q$  Hermitian  $\Rightarrow K(\frac{2\pi v}{T})$  is real for all integers  $v$ ,  
 $Q$  real  $\rightarrow K(\frac{2\pi v}{T}) = K^*(\frac{2\pi(T-v)}{T})$ .

In the sequel we shall call  $Q$  the circulant matrix generated by  $K(\omega)$ .

Lemma 2. Let  $R(\tau)$ ,  $\tau = \dots -1, 0, 1$ , be a doubly infinite sequence of real numbers with

$$\sum_{\tau=-\infty}^{\infty} |\tau| |R(\tau)| = \theta < \infty,$$

let

$$f(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} R(\tau) e^{-i\omega\tau}$$

and for fixed  $T$ , let  $D_f$  be the  $T \times T$  diagonal matrix with  $r$ -th entry  $f(\frac{2\pi r}{T})$ . Let  $W$  be as in Lemma 1, and let

$$\begin{aligned} \tilde{\Sigma} &= \{\tilde{\sigma}_{\mu\nu}, \mu, \nu = 1, 2, \dots, T\} = 2\pi W D_f W^* \\ \Sigma &= \{\sigma_{\mu\nu}, \mu, \nu = 1, 2, \dots, T\} \quad \sigma_{\mu\nu} = R(\mu-\nu). \end{aligned}$$

For any matrices of the same dimensions, define

$$\varphi(A-B) = \sum_{\mu, \nu} |a_{\mu\nu} - b_{\mu\nu}|, \quad A = \{a_{\mu\nu}\}, \quad B = \{b_{\mu\nu}\}.$$

where  $W$  is the unitary matrix defined in Lemma 1 and  $D_{ij}$  is the  $T \times T$  diagonal matrix with  $r, r$ -th entry  $f_{ij}(\frac{2\pi r}{T})$ . Since  $f_{ii}(\frac{2\pi r}{T}) = f_{ii}(\frac{2\pi(T-r)}{T})$  and  $f_{ij}(\frac{2\pi r}{T}) = f_{ij}(\frac{2\pi(T-r)}{T}) = f_{ji}^*(\frac{2\pi r}{T})$ , the  $2T \times 2T$  matrix  $\tilde{\Sigma}$  given by

$$(3.2) \quad \tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} \\ \tilde{\Sigma}_{21} & \tilde{\Sigma}_{22} \end{pmatrix}$$

is real symmetric. It is readily verified that upper and lower bounds for the eigenvalues of  $\tilde{\Sigma}$  are given by  $\Lambda = \max_{\omega} \Lambda(\omega)$  and  $\lambda = \min_{\omega} \lambda(\omega)$  where  $\lambda(\omega)$  and  $\Lambda(\omega)$  are the smallest and largest eigenvalues of  $F(\omega)$ .

Lemma 3. Let  $(\tilde{X}_1: \tilde{X}_2)$  be a  $2T$  dimensional zero mean Gaussian random vector with covariance matrix  $\tilde{\Sigma}$  defined by (3.2). Let  $K(\omega)$  be a real function of  $\omega$  defined on  $[0, 2\pi]$  and let  $Q$  be the  $T \times T$  Hermitian circulant matrix generated by  $K(\omega)$ . Let  $\tilde{X}$  be the  $2 \times T$  matrix  $\tilde{X} = \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}$ . Then the random matrix

$$\hat{F}_{\tilde{X}} = \frac{1}{T} \tilde{X} Q \tilde{X}'$$

is distributed as the random matrix

$$(3.3) \quad \frac{2\pi}{T} \sum_{r=1}^T z_r z_r^{*'} K\left(\frac{2\pi r}{T}\right)$$

where, for each  $r$ ,  $z_r$  is a 2 dimensional complex normal [3] random (column) vector with complex covariance

$$E z_r z_r^{*'} = F\left(\frac{2\pi r}{T}\right), \quad r = 1, 2, \dots, T$$

with  $z_r \equiv z_{T-r}^*$  and  $z_r$  and  $z_s$  independent for  $s \neq r$  or  $T-r$ .

$$\underline{F}(\omega_0) = \underline{F}_{T,n}(\omega_0) = \frac{1}{2n+1} \sum_{r=j_0-n}^{j_0+n} F\left(\frac{2\pi r}{T}\right), \quad \omega_0 = \frac{2\pi j_0}{T}.$$

Let  $\lambda, \Lambda$  defined above satisfy  $0 < \lambda \leq \Lambda < \infty$ , and suppose

$\sum_{i,j=1}^2 \sum_{\tau=-\infty}^{\infty} |\tau| |R_{ij}(\tau)| = \theta < \infty$ . Then it is possible to construct an  $\hat{F}(\omega_0)$  on the same sample space as  $\hat{F}_{\tilde{X},T,n}(\omega_0)$  so that

$$(3.5) \quad E \operatorname{Tr}(\hat{F}_{\tilde{X},T,n}(\omega_0) - \hat{F}(\omega_0)) (\hat{F}_{\tilde{X},T,n}(\omega_0) - \hat{F}(\omega_0))^* \leq 6 \frac{\Lambda}{\lambda} \theta^2 \frac{(2n+1)}{T^2}$$

Proof. From Lemma 3 and (3.4)

$$\begin{aligned} \hat{F}_{\tilde{X},T,n}(\omega_0) &= \frac{1}{2n+1} \sum_{r=j_0-n}^{j_0+n} z_r z_r^* \\ &\sim \frac{1}{2n+1} \sum_{r=j_0-n}^{j_0+n} F^{1/2}\left(\frac{2\pi r}{T}\right) \xi_r \xi_r^* F^{1/2}\left(\frac{2\pi r}{T}\right) \end{aligned}$$

where the  $\{\xi_r = F^{-1/2}(\frac{2\pi r}{T}) z_r\}$  are independent complex normal random vectors with complex covariance  $E \xi_r \xi_r^* = I_{2 \times 2}$ . (Here, as in the sequel, for  $A$  Hermitian or symmetric,  $A^{1/2}$  is the Hermitian or symmetric square root).

Hence

$$\hat{F}_{\tilde{X},T,n}(\omega_0) \sim \hat{F}(\omega_0) + \frac{1}{2n+1} \sum_{r=j_0-n}^{j_0+n} G_r$$

where

$$(3.6) \quad \hat{F}(\omega_0) = \frac{1}{2n+1} F^{1/2}(\omega_0) \left( \sum_{r=j_0-n}^{j_0+n} \xi_r \xi_r^* \right) F^{1/2}(\omega_0) \sim W_c(F(\omega_0), 2, 2n+1)$$

and

$$G_r = F^{1/2}\left(\frac{2\pi r}{T}\right) \xi_r \xi_r^* F^{1/2}\left(\frac{2\pi r}{T}\right) - F^{1/2}(\omega_0) \xi_r \xi_r^* F^{1/2}(\omega_0)$$

Using the facts that  $E \operatorname{Tr} G_r G_s^{*'} = \operatorname{Tr}(F(\frac{2\pi r}{T}) - \underline{F}(\omega_0))(F(\frac{2\pi s}{T}) - \underline{F}(\omega_0))$ ,  $r \neq s$

$$\frac{1}{(2n+1)^2} \operatorname{Tr} \sum_{r=j_0-n}^{j_0+n} \sum_{s=j_0-n}^{j_0+n} (F(\frac{2\pi r}{T}) - \underline{F}(\omega_0))(F(\frac{2\pi s}{T}) - \underline{F}(\omega_0))^{*'} = 0$$

$$\frac{1}{(2n+1)^2} \operatorname{Tr} \sum_{r=j_0-n}^{j_0+n} (F(\frac{2\pi r}{T}) - \underline{F}(\omega_0))(F(\frac{2\pi r}{T}) - \underline{F}(\omega_0))^{*'} \geq 0$$

gives the inequality

$$(3.7) E \operatorname{Tr} \left( \frac{1}{2n+1} \sum_{r=j_0-n}^{j_0+n} G_r \right) \left( \frac{1}{2n+1} \sum_{r=j_0-n}^{j_0+n} G_r \right)^{*'} \leq \frac{1}{(2n+1)^2} \sum_{r=j_0-n}^{j_0+n} E \operatorname{Tr}(G_r G_r^{*'}) .$$

Observe now that

$$\begin{aligned} \phi(F(\frac{2\pi r}{T}) - \underline{F}(\omega_0)) &\leq \frac{1}{2\pi(j_0-n)/T \leq \omega \leq 2\pi(j_0+n)/T} \sum_{i,j=1}^2 |f_{ij}(\frac{2\pi r}{T}) - f_{ij}(\omega)| \\ &\leq \theta \frac{(2n+1)}{T}, \text{ since, for } |\omega - \frac{2\pi r}{T}| \leq 2\pi \frac{(2n+1)}{T}, \\ |f_{ij}(\frac{2\pi r}{T}) - f_{ij}(\omega_0)| &\leq \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} |R_{ij}(\tau)| |1 - e^{(\frac{2\pi ir}{T} - \omega)\tau}| \leq \frac{(2n+1)}{T} \sum_{\tau=-\infty}^{\infty} |\tau| |R_{ij}(\tau)|. \end{aligned}$$

Lemma A.4 of the appendix then yields the inequality

$$(3.8) \operatorname{Tr}(G_r G_r^{*'}) \leq [\max \text{ eigenvalue } \xi_r \xi_r^{*'}]^2 \frac{\Lambda(2n+1)^2}{\lambda} \theta^2 \leq (\|\xi_r\|^2)^2 \frac{\Lambda(2n+1)^2}{\lambda} \theta^2.$$

Observing that  $E\|\xi_r\|^4 = 6$ , and putting together (3.7) and (3.8) gives the result.

#### 4. Asymptotic Behavior of Quadratic Forms in Stationary Gaussian Processes.

We first prove a lemma which puts a bound on the mean square difference between a general quadratic form in a normal vector  $Y$  and the same quadratic form in  $\tilde{Y}$ , a normal random vector approximating  $Y$ .



Lemma 5. Let  $Y \sim N(0, S)$ , let  $A$  be any (real or complex) quadratic form with  $q^2$  the largest eigenvalue of  $AA^{*}$ . Let  $\tilde{S}$  be a symmetric positive definite matrix of the same dimensions as  $S$ , let  $\varphi(S-\tilde{S}) \leq \theta$ , and let  $\Lambda$  and  $\lambda$  be common upper and lower bounds for the eigenvalues of  $S$  and  $\tilde{S}$ ,  $0 < \lambda < \Lambda < \infty$ . Let  $\tilde{Y} = YS^{-1/2} \tilde{S}^{1/2}$ . Then

$$(4.1) \quad E|YAY' - \tilde{Y}\tilde{A}\tilde{Y}'|^2 \leq (1 + 2 \frac{\Lambda}{\lambda}) q^2 \theta^2.$$

Proof.

$$|YAY' - \tilde{Y}\tilde{A}\tilde{Y}'|^2 = |Y(A - S^{-1/2} \tilde{S}^{1/2} A \tilde{S}^{1/2} S^{-1/2})Y'|^2.$$

Letting  $H = A - S^{-1/2} \tilde{S}^{1/2} A \tilde{S}^{1/2} S^{-1/2}$ , we have, using the relations between 4th order mixed moments of Gaussian random variables

$$(4.2) \quad E|YHY'|^2 = (\text{Tr } HS)^2 + 2 \text{Tr } HSH'S,$$

$$(4.3) \quad (\text{Tr } HS)^2 = (\text{Tr } A(S-\tilde{S}))^2 \leq [\text{largest absolute entry of } A \times \varphi(S-\tilde{S})]^2 \leq q^2 \theta^2,$$

$$(4.4) \quad \begin{aligned} \text{Tr } HSH'S &= \text{Tr}(A - S^{-1/2} \tilde{S}^{1/2} A \tilde{S}^{1/2} S^{-1/2})S(A - S^{-1/2} \tilde{S}^{1/2} A \tilde{S}^{1/2} S^{-1/2})'S \\ &= \text{Tr}(S^{1/2} A S^{1/2} - \tilde{S}^{1/2} A \tilde{S}^{1/2})(S^{1/2} A S^{1/2} - \tilde{S}^{1/2} A \tilde{S}^{1/2})'. \end{aligned}$$

Using Lemma A.4 of the appendix on the right-hand side of (4.4) yields the inequality

$$(4.5) \quad \text{Tr } HSH'S \leq \frac{\Lambda}{\lambda} q^2 \theta^2.$$

Combining (4.2), (4.3) and (4.5) gives the result.

Theorem 1. Let  $X(t)$ ,  $t = \dots -1, 0, 1, \dots$  be a two dimensional stationary zero mean Gaussian stochastic process possessing a spectral density matrix  $F(\omega)$ . Suppose:

$$1. \quad 0 < \lambda < \text{eigenvalues of } F(\omega) < \Lambda < \infty ;$$

$$2. \quad \sum_{i,j=1}^2 \sum_{\tau=-\infty}^{\infty} |\tau| R_{ij}(\tau) = \theta < \infty .$$

For each  $T$ , let  $X$  be defined as in Section 1. Let  $\tilde{X}_1$  and  $\tilde{X}_2$  be  $T$ -dimensional zero mean Gaussian (row) vectors with  $E \tilde{X}_i \tilde{X}_j' = \tilde{\Sigma}_{ij}$ ,

$i, j = 1, 2$ , defined in (3.1), let  $\tilde{X} = \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}$  and let  $Q$  be any  $T \times T$

quadratic form with largest absolute eigenvalue  $\leq q$ .

Let

$$\hat{F}_{X,Q} = \frac{1}{T} X Q X'$$

and

$$\hat{F}_{\tilde{X},Q} = \frac{1}{T} \tilde{X} Q \tilde{X}' .$$

Then for each  $T$ , it is possible to construct an  $\tilde{X}$  independent of  $Q$  on the sample space of  $X$  such that

$$(4.6) \quad E \text{Tr}(\hat{F}_{X,Q} - \hat{F}_{\tilde{X},Q})(\hat{F}_{X,Q} - \hat{F}_{\tilde{X},Q})^{*'} \leq 4(1 + 2 \frac{\Lambda}{\lambda}) (\frac{q^2}{T^2}) (2\theta)^2 .$$

Proof. Let

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} ,$$

where the  $\Sigma_{ij}$  are as in Section 1. It follows, by application of an argument directly analogous to [4], p. 64, that the eigenvalues of  $\Sigma$  are bounded above and below by  $\Lambda$  and  $\lambda$ .

Let the  $2T$  dimensional vector  $(\tilde{X}_1: \tilde{X}_2)$  be given by

$$(\tilde{X}_1: \tilde{X}_2) = (X_1: X_2) \Sigma^{-1/2} \tilde{\Sigma}^{1/2},$$

then  $E \tilde{X}_i' \tilde{X}_j = \tilde{\Sigma}_{ij}$ . Applying Lemma 2 to each of the blocks  $\Sigma_{ij} = \tilde{\Sigma}_{ij}$  of  $\Sigma = \tilde{\Sigma}$  gives

$$\phi(\Sigma - \tilde{\Sigma}) \leq 2 \sum_{i,j=1}^2 \sum_{\tau=-\infty}^{\infty} |\tau| |R_{ij}(\tau)| = 2\theta.$$

Now

$$\begin{aligned} & E \text{Tr}(\hat{F}_{X,Q} - \hat{F}_{\tilde{X},Q})(\hat{F}_{X,Q} - \hat{F}_{\tilde{X},Q})^* \\ &= \frac{1}{T^2} E \sum_{i,j=1}^2 |(X_1: X_2) Q_{ij} (X_1: X_2)' - (\tilde{X}_1: \tilde{X}_2) Q_{ij} (\tilde{X}_1: \tilde{X}_2)'|^2 \end{aligned}$$

where  $Q_{ij}$  is the  $2T \times 2T$  matrix with  $Q$  in the  $i,j$ -th block (of dimension  $T \times T$ ) and zeroes elsewhere; with the largest eigenvalue of  $Q_{ij} Q_{ij}^* \leq q^2$ . Applying Lemma 5 with  $Y = (X_1: X_2)$ ,  $S = \Sigma$  and  $\tilde{S} = \tilde{\Sigma}$  gives the result.

An application of this theorem will allow us to show that  $M$  sample spectral density matrices, calculated at  $M$  appropriately spaced frequencies converge jointly in mean square to  $M$  independent complex Wishart matrices, even though  $M$  be large. Let  $\hat{F}_{X,T,n}(\omega_\ell) = \frac{1}{T} X Q_{T,n}(\omega_\ell) X'$ ,  $\ell = 1, 2, \dots, M$  be the sample spectral density matrices formed from the average of  $2n+1$  periodograms centered at  $\omega_\ell = \frac{2\pi j_\ell}{T}$ , where the  $j_\ell$  are chosen so that the  $M$  sets of integers  $\{j_\ell + j, j = 0, \pm 1, \pm 2, \dots, \pm n\}$ ,  $\ell = 1, 2, \dots, M$ , are disjoint, and

$$0 < \frac{2\pi(j_\ell - n)}{T} < \frac{2\pi(j_\ell + n)}{T} \leq \pi. \text{ We have necessarily } (2n+1) M \leq \frac{T}{2}.$$

It will be convenient to define  $\tilde{X}$ , a  $2 \times T$  random matrix by  $\tilde{Z} = \tilde{X} W$ , where the  $r^{\text{th}}$  column  $\tilde{z}_r$  of  $\tilde{Z}$  is given by

$$(4.7) \quad \tilde{z}_r = F^{1/2}(\omega_\ell) F^{-1/2}\left(\frac{2\pi r}{T}\right) z_r, \quad j_\ell - n \leq r \leq j_\ell + n, \quad \omega_\ell = \frac{2\pi j_\ell}{T}$$

$$\tilde{z}_s = \tilde{z}_r^* \quad s = T - r$$

$$\tilde{z}_r = z_r \quad \text{otherwise}$$

with  $F(\omega_\ell) = \frac{1}{(2n+1)} \sum_{r=j_\ell-n}^{j_\ell+n} F\left(\frac{2\pi r}{T}\right)$ , and  $z_r$  defined from  $\tilde{X}$  as in the proof of Lemma 3. Then  $\hat{F}_{\tilde{X}, T, n}(\omega_\ell) = \frac{1}{T} \tilde{X} Q_{T, n}(\omega_\ell) \tilde{X}^*$  for each  $\ell = 1, 2, \dots, M$  are constructed exactly as the  $\hat{F}(\omega_0)$  of (3.6) and, by Lemmas 3 and 4 are a set of independent  $W_c(F(\omega_\ell), 2, 2n+1)$  random matrices.

We have the following

**Corollary 1** Under the conditions of the theorem,  $\{\hat{F}_{\tilde{X}, T, n}(\omega_\ell), \ell = 1, 2, \dots, M\}$ , where  $\omega_\ell$  are chosen as above, jointly converge in mean square to  $M$  independent  $W_c(F(\omega_\ell), 2, 2n+1)$  matrices, as  $n, M, T \rightarrow \infty$ , provided only that  $\log_2 M \leq n$ . More precisely, let  $\{\hat{F}_{\tilde{X}, T, n}(\omega_\ell)\}$  be the  $M$  independent complex Wishart matrices defined above. Then

$$(4.8) \quad E \sum_{\ell=1}^M \text{Tr} (\hat{F}_{\tilde{X}, T, n}(\omega_\ell) - \hat{F}_{\tilde{X}, T, n}^W(\omega_\ell)) (\hat{F}_{\tilde{X}, T, n}(\omega_\ell) - \hat{F}_{\tilde{X}, T, n}^W(\omega_\ell))^* \\ \leq (12 \frac{\Lambda}{\lambda} \theta^2) \frac{(2n+1)M}{T^2} + 32(1 + 2 \frac{\Lambda}{\lambda}) \theta^2 \frac{\log_2 M}{(2n+1)^2} \rightarrow 0$$

Proof. Using lemma A.1, the left hand side of (4.8) is less than

$$(4.9) \quad (2 \sum_{\ell=1}^M E \text{Tr} (\hat{F}_{\tilde{X}, T, n}(\omega_\ell) - \hat{F}_{\tilde{X}, T, n}^W(\omega_\ell)) (\hat{F}_{\tilde{X}, T, n}(\omega_\ell) - \hat{F}_{\tilde{X}, T, n}^W(\omega_\ell))^*) \\ + (2 \sum_{\ell=1}^M E \text{Tr} (\hat{F}_{\tilde{X}, T, n}^W(\omega_\ell) - \hat{F}_{\tilde{X}, T, n}^W(\omega_\ell)) (\hat{F}_{\tilde{X}, T, n}^W(\omega_\ell) - \hat{F}_{\tilde{X}, T, n}^W(\omega_\ell))^*)$$

By lemma 4, the second term in (4.9) is bounded by  $12 \frac{\Lambda}{\lambda} \theta^2 \frac{(2n+1)M}{T^2}$ .

To bound the first term, let  $Q = Q_{T,n}(s_1, s_2, \dots, s_M) = \sum_{\ell=1}^M s_{\ell} Q_{T,n}(\omega_{\ell})$ .

Then

$$(4.10) \quad \sum_{\ell=1}^M s_{\ell} (\hat{F}_{X,T,n}(\omega_{\ell}) - \hat{F}_{X,T,n}^{\omega}(\omega_{\ell})) = \frac{1}{T} (X Q X' - \tilde{X} Q \tilde{X}').$$

Since, by the choice of  $\omega_{\ell}$ ,  $\{\frac{2n+1}{T} Q_{T,n}(\omega_{\ell}), \ell = 1, 2, \dots, M\}$  are a family of orthogonal projections, the largest eigenvalue of  $QQ^*$  is equal to

$\max_{\ell} |s_{\ell}|^2 (\frac{T}{2n+1})^2$ . Applying Theorem 1 to (4.10) then gives

$$(4.11) \quad \sum_{\ell=1}^M \sum_{k=1}^M s_{\ell} s_k E \text{Tr}(\hat{F}_{X,T,n}(\omega_{\ell}) - \hat{F}_{X,T,n}^{\omega}(\omega_{\ell}))(\hat{F}_{X,T,n}(\omega_k) - \hat{F}_{X,T,n}^{\omega}(\omega_k))^* \\ \leq 4(1 + \frac{2\lambda}{\lambda})(2\theta)^2 \frac{1}{(2n+1)^2} \max_{\ell} |s_{\ell}|^2$$

Now let  $A$  be the (non-negative definite)  $M \times M$  matrix with  $\ell k^{\text{th}}$  element

$a_{\ell k}$  given by  $a_{\ell k} = E \text{Tr}(\hat{F}_{X,T,n}(\omega_{\ell}) - \hat{F}_{X,T,n}^{\omega}(\omega_{\ell}))(\hat{F}_{X,T,n}(\omega_k) - \hat{F}_{X,T,n}^{\omega}(\omega_k))^*$

Using Lemma A.5, we see that (4.11) implies that

$$\text{Tr } A = \sum_{\ell=1}^M E \text{Tr}(\hat{F}_{X,T,n}(\omega_{\ell}) - \hat{F}_{X,T,n}^{\omega}(\omega_{\ell}))(\hat{F}_{X,T,n}(\omega_{\ell}) - \hat{F}_{X,T,n}^{\omega}(\omega_{\ell}))^* \leq \\ 4(1 + \frac{2\lambda}{\lambda})(2\theta)^2 \frac{\log_2 M}{(2n+1)^2}.$$

Let  $\{R(\tau) = \frac{1}{T} X U_{\tau} X', \tau = 0, 1, 2, \dots, L < T\}$  be the sample circularized autocorrelation matrices, where  $U_{\tau}$  is the symmetric circulant matrix with  $\frac{1}{2}$  down the  $\tau$ -th and  $T - \tau$ th diagonal and zeroes elsewhere on and above diagonal

(with  $U_0 = I$ ). As is well known (and obvious from Lemma 3),

$\{\hat{R}(\tau) = \frac{1}{T} \tilde{X} U_{\tau} \tilde{X}', \tau = 0, 1, 2, \dots, L\}$  are jointly distributed as the  $L$

random matrices

$$\frac{1}{T} \sum_{r=1}^T \cos\left(\frac{2\pi r \tau}{T}\right) z_r z_r^*, \quad \tau = 1, 2, \dots, L$$

where  $z_r = z_{T-r}^*$ ,  $z_r$  and  $z_s$  are independent, for  $s \neq r$  or  $T - r$ , complex normal vectors with  $E z_r z_r^* = F\left(\frac{2\pi r}{T}\right)$ . We have

### Corollary 2

$$E \operatorname{Tr} (\hat{R}(\tau) - \hat{R}_0(\tau))(\hat{R}(\tau) - \hat{R}_0(\tau))^* \leq 4(1 + 2 \frac{\Lambda}{\lambda}) (2\theta)^2 \frac{1}{T^2}, \quad \tau = 0, 1, 2, \dots, L$$

### 5. Applications To Hypothesis Testing For Stationary Time Series.

Suppose we observed a  $2 \times T$  random matrix distributed as  $\tilde{X}$ . Recall that  $\tilde{X}$  is an approximation to the random matrix  $X$ , which is a record of length  $T$  of the process  $X(t)$  where the approximation proceeded in two steps, first by circularizing  $X$  to get  $\tilde{X}$ , and then by replacing  $F(\omega)$  by  $\underline{F}(\omega) = \underline{F}(\omega_\ell)$  for  $\frac{2\pi j_\ell - n}{T} \leq \omega \leq \frac{2\pi j_\ell + n}{T}$ . (Here we let the union of the intervals  $\{[\frac{2\pi j_\ell - n}{T}, \frac{2\pi j_\ell + n}{T}]\}$  cover the points  $\{\frac{2\pi j}{T}, j = 1, 2, \dots, [\frac{T}{2}]\}$ ). The hypothesis that the 2 rows of  $\tilde{X}$  are independent is equivalent to the hypothesis that  $\underline{F}(\omega)$  is diagonal, all  $\omega$ , which represents an approximation to the hypothesis that the two time series  $X_1(t)$  and  $X_2(s)$  are independent, i.e.  $|f_{12}(\omega)|^2 = 0$ , all  $\omega$ . Considering  $\tilde{X}$ , the likelihood ratio statistic for the hypothesis  $\underline{F}(\omega) = \text{diagonal, all } \omega$  may be readily gotten by examining the well known results [2] in the analogous situation of testing for the diagonality of the covariance matrix of a (real) normal random vector. It is

$$\lambda = \left[ \frac{\sum_{\ell=1}^M |\hat{F}_{\tilde{X}, T, n}(\omega_\ell)|^2}{\hat{f}_{11}(\omega_\ell) \hat{f}_{22}(\omega_\ell)} \right]^{\frac{1}{M}} = \left[ \sum_{\ell=1}^M (1 - \hat{w}_\ell(\omega_\ell)) \right]^{\frac{1}{M}}$$

where

$$\hat{W}(\omega_\ell) = \frac{|\hat{f}_{12}(\omega_\ell)|^2}{\hat{f}_{11}(\omega_\ell) \hat{f}_{22}(\omega_\ell)}, \quad \{\hat{f}_{ij}(\omega_\ell)\}_{i,j=1,2} = \hat{F}_{X,T,n}^{(2)}(\omega_\ell)$$

This suggests using  $\lambda$  to test the hypothesis of independence  $X_1(t)$  and  $X_2(s)$ ,

where

$$\lambda = \left[ \prod_{\ell=1}^M (1 - \hat{W}(\omega_\ell)) \right]^{\frac{1}{M}},$$

$$\hat{W}(\omega_\ell) = \frac{|\hat{f}_{12}(\omega_\ell)|^2}{\hat{f}_{11}(\omega_\ell) \hat{f}_{22}(\omega_\ell)}, \quad \{\hat{f}_{ij}(\omega_\ell)\}_{i,j=1,2} = \hat{F}_{X,T,n}^{(2)}(\omega_\ell)$$

Under the null hypothesis,  $\hat{W}(\omega_k)$ ,  $k = 1, 2, \dots, M$  are distributed as  $M$  independent  $\beta_{1,2n}$  random variables, hence  $\text{Var } -2n\sqrt{M} \log \lambda = 1$ , and under the alternative, for large  $M, n$ ,  $\text{Var } -\sqrt{nM} \log \lambda \rightarrow \text{constant}$ . (See [3] for the density of  $\hat{W}(\omega_k)$ ).

We have the following

Corollary 3. As  $n, M, T \rightarrow \infty$ , in such a way that  $\frac{\log_2 M}{n} \rightarrow 0$

$$E \left| \sqrt{nM} (\log \lambda - \log \lambda) \right| \rightarrow 0$$

If  $|f_{12}(\omega)| \neq 0$ , then

$$E \left| n\sqrt{M} (\log \lambda - \log \lambda) \right| \rightarrow 0$$

Proof.

Using the fact that  $|\log(1-\mu) - \log(1-\nu)| \leq |\mu-\nu| \left| \frac{1}{(1-\mu)} + \frac{1}{(1-\nu)} \right|$  for  $0 \leq \mu, \nu < 1$ , and rewriting  $\hat{W}(\omega_\ell) - \hat{W}(\omega_\ell)$  in terms of the entries of  $\hat{F}_{X,T,n}^{(2)}(\omega_\ell)$ ,  $\hat{F}_{X,T,n}^{(2)}(\omega_\ell)$  and  $(\hat{F}_{X,T,n}^{(2)}(\omega_\ell) - \hat{F}_{X,T,n}^{(2)}(\omega_\ell))$ , we may obtain

$$\begin{aligned}
 E \sqrt{n M} |\log \hat{\lambda} - \log \lambda| &\leq \sqrt{\frac{n}{M}} \sum_{\ell=1}^M E |\log(1 - \hat{W}(\omega_\ell)) - \log(1 - \hat{W}(\omega_\ell))| \\
 &\leq \sqrt{\frac{n}{M}} \sum_{\ell=1}^M E [g(\omega_\ell) \left( \sum_{i,j=1}^2 |\hat{f}_{ij}(\omega_\ell) - \hat{f}_{ij}(\omega_\ell)| \right)] \\
 (5.1) \quad &\leq \text{const} \sqrt{\frac{n}{M}} \sum_{\ell=1}^M [E g^2(\omega_\ell)]^{1/2} \\
 &\quad [E \text{Tr}(\hat{F}_{X,T,n}(\omega_\ell) - \hat{F}_{X,T,n}^{\sim}(\omega_\ell))(\hat{F}_{X,T,n}(\omega_\ell) - \hat{F}_{X,T,n}^{\sim}(\omega_\ell))^*]^{1/2}
 \end{aligned}$$

where  $E g^2(\omega_\ell)$  is bounded by a constant depending only on  $F(\omega)$ . Observe that an inequality of the type (5.1) applies to a fairly general class of  $\lambda$ 's formed from products of functions of  $\hat{F}_{X,T,n}(\omega_\ell)$ . This results in

$$\begin{aligned}
 E \sqrt{n M} |\log \hat{\lambda} - \log \lambda| \\
 \leq \text{const} \sqrt{n} \left[ \sum_{\ell=1}^M E \text{Tr}(\hat{F}_{X,T,n}(\omega_\ell) - \hat{F}_{X,T,n}^{\sim}(\omega_\ell))(\hat{F}_{X,T,n}(\omega_\ell) - \hat{F}_{X,T,n}^{\sim}(\omega_\ell))^* \right]^{1/2}
 \end{aligned}$$

which, by Corollary 1 is less than  $\text{const} \sqrt{n} \left( \frac{\log_2 M}{(2n+1)^2} + \frac{M(2n+1)}{T^2} \right)^{1/2} \rightarrow 0$  as

$M, n \rightarrow \infty$  in such a way that  $\frac{\log_2 M}{n} \rightarrow 0$ .

When the null hypothesis is true,  $|\hat{f}_{12}(\omega_\ell)| + |\hat{f}_{12}(\omega_\ell)|$  may be factored out of  $g(\omega_\ell)$  to get an expression of the form



$$E n \sqrt{M} |\log \lambda_{\hat{\omega}} - \log \lambda|$$

$$\leq \text{constant} \frac{n}{\sqrt{M}} \sum_{\ell=1}^M (E h^4(\omega_{\ell}))^{1/4} (E |\hat{f}_{12}(\omega_{\ell})|^4 + E |\hat{f}_{12}(\omega_{\ell})|^4)^{1/4} \times$$

$$[E \text{Tr}(\hat{F}_{X,T,n}(\omega_{\ell}) - \hat{F}_{X,T,n}^{\omega}(\omega_{\ell}))(\hat{F}_{X,T,n}(\omega_{\ell}) - \hat{F}_{X,T,n}^{\omega}(\omega_{\ell}))^*]^{1/2}$$

where  $E h^4(\omega_{\ell})$  is bounded by a constant depending only on  $F(\omega)$ . Using the facts that  $\hat{f}_{12}(\omega_{\ell})$  and  $\hat{f}_{12}(\omega_{\ell})$  are quadratic forms in normal random variables, and for such quadratic forms  $t$ ,  $E t^4 \leq c (E t^2)^2$ , where  $c$  is a universal constant, and under the null hypothesis,  $E |\hat{f}_{12}(\omega_{\ell})|^2$  and  $E |\hat{f}_{12}(\omega_{\ell})|^2$  are bounded by a constant  $\times \frac{1}{2n+1}$ , the result is

$$E n \sqrt{M} |\log \lambda_{\hat{\omega}} - \log \lambda|$$

$$\leq \text{constant} \cdot \frac{n}{(2n+1)^{1/2}} \left( \frac{\log_2 M}{(2n+1)^2} + \frac{M(2n+1)}{T^2} \right)^{1/2} \rightarrow 0.$$

# Appendix

In Lemmas A.1-A.3, A and B are strictly positive definite matrices,  $U, U_i$  and V are any square matrices, and  $\lambda_Z, \Lambda_Z$  are the smallest and largest eigenvalues of the matrix Z. The fact that  $\lambda_A \text{Tr } B \leq \text{Tr } AB \leq \Lambda_A \text{Tr } B$  is repeatedly used.

## Lemma A.1

$$\text{Tr} \left( \sum_{j=1}^N U_j \right) \left( \sum_{j=1}^N U_j \right)^{*'} \leq N \text{Tr} \left( \sum_{j=1}^N U_j U_j^{*'} \right) .$$

## Proof.

$$\begin{aligned} \text{Tr} \left( \sum_{j=1}^N U_j \right) \left( \sum_{j=1}^N U_j \right)^{*'} &= \text{Tr} \sum_{j=1}^N \sum_{k=1}^N U_j U_k^{*'} \\ &\leq \text{Tr} \sum_{j=1}^N \sum_{k=1}^N \frac{1}{2} (U_j U_j^{*'} + U_k U_k^{*'}) \\ &= N \text{Tr} \left( \sum_{j=1}^N U_j U_j^{*'} \right) . \end{aligned}$$

## Lemma A.2

$$\text{Tr}(A-B)^2 \leq \frac{1}{(\lambda_A + \lambda_B)^2} \text{Tr}(A^2 - B^2)^2 .$$

## Proof.

$$(\lambda_A + \lambda_B) \text{Tr}(A-B)^2 \leq \text{Tr}(A-B)^2 (A+B) = \text{Tr}(A-B)(A+B)(A-B)$$

$$(\lambda_A + \lambda_B) \text{Tr}(A-B)(A+B)(A-B) \leq \text{Tr}(A-B)(A+B)(A-B)(A+B)$$

giving

$$(\lambda_A + \lambda_B)^2 \text{Tr}(A-B)^2 \leq \text{Tr}(A-B)(A+B)(A-B)(A+B) .$$

$$\begin{aligned}
 \text{Tr}(A-B)(A+B)(A-B)(A+B) &= \text{Tr}(A^2 - BA + AB - B^2)(A^2 - BA + AB - B^2) \\
 &= \text{Tr}[A^4 - A^2BA + A^3B - A^2B^2 - BA^3 + BABA - BAAB + BAB^2 + ABA^2 \\
 &\quad - ABBA + ABAB - AB^3 - B^2A^2 + B^3A - B^2AB + B^4] \\
 &= \text{Tr}[A^4 - 4B^2A^2 + 2BABA + B^4] .
 \end{aligned}$$

Now,

$$\text{Tr } BABA \leq \text{Tr } A^2B^2 ,$$

hence

$$\text{Tr}[A^4 - 4B^2A^2 + 2BABA + B^4] \leq \text{Tr}[A^4 - 2 \text{Tr } A^2B^2 + B^4] = \text{Tr}[A^2 - B^2]^2 .$$

Combining these last inequalities gives the lemma. It is easy to see that equality is obtained for  $A = aI$ ,  $B = bI$ .

#### Lemma A.3

$$\text{Tr}(U-V)(U-V)^{*'} \leq [\varphi(U-V)]^2 .$$

Proof.

$$\text{Tr}(U-V)(U-V)^{*'} = \sum_{\mu, \nu} (u_{\mu\nu} - v_{\mu\nu})^2 \leq \left[ \sum_{\mu, \nu} |u_{\mu\nu} - v_{\mu\nu}| \right]^2 = [\varphi(U-V)]^2 .$$

Lemma A.4 Let  $A$  be any (real or complex) quadratic form with  $q^2$  the largest eigenvalue of  $AA^{*'}$ . Let  $S, \tilde{S}$  be strictly positive (real or complex) matrices of the same dimension as  $A$  and let  $0 < \lambda \leq \Lambda < \infty$  be common lower and upper bounds for the eigenvalues of  $S$  and  $\tilde{S}$ , and suppose  $\varphi(S - \tilde{S})$ , defined in Lemma 2 satisfies  $\varphi(S - \tilde{S}) \leq \theta$ . Then

$$\text{Tr}(S^{1/2}AS^{1/2} - \tilde{S}^{1/2}A\tilde{S}^{1/2})(S^{1/2}AS^{1/2} - \tilde{S}^{1/2}A\tilde{S}^{1/2})^{*'} \leq \frac{\Lambda}{\lambda} q^2 \theta^2 .$$

Proof.

$$\begin{aligned}
 &\text{Tr}(S^{1/2}AS^{1/2} - \tilde{S}^{1/2}A\tilde{S}^{1/2})(S^{1/2}AS^{1/2} - \tilde{S}^{1/2}A\tilde{S}^{1/2})^{*'} \\
 &= \text{Tr}[(S^{1/2}\tilde{S}^{1/2})AS^{1/2} + \tilde{S}^{1/2}A(S^{1/2} - \tilde{S}^{1/2})][(S^{1/2}\tilde{S}^{1/2})AS^{1/2} + \tilde{S}^{1/2}A(S^{1/2} - \tilde{S}^{1/2})]^{*'} ,
 \end{aligned}$$

which by Lemma A.1 is less than

$$2 \operatorname{Tr}(S^{1/2} \tilde{S}^{1/2})(S^{1/2} \tilde{S}^{1/2})^{*'} (ASA^{*'} + A\tilde{S}A^{*'}) \\ \leq 4q^2 \Lambda \operatorname{Tr}(S^{1/2} \tilde{S}^{1/2})(S^{1/2} \tilde{S}^{1/2})^{*'} .$$

Lemmas A.2 and A.3 give

$$\operatorname{Tr}(S^{1/2} \tilde{S}^{1/2})(S^{1/2} \tilde{S}^{1/2})^{*'} \\ \leq \frac{1}{4\lambda} \operatorname{Tr}(S - \tilde{S})(S - \tilde{S})^{*'} \leq \frac{1}{4\lambda} [\varphi(S - \tilde{S})]^2 = \frac{1}{4\lambda} \theta^2 ,$$

which gives the result.

Lemma A.5 Let  $A$  be a non-negative definite  $M \times M$  matrix, and suppose it is known that, for any  $s = (s_1, s_2, \dots, s_M)$ ,  $s A s' \leq \max_{\ell} |s_{\ell}|^2 c^2$ . Then, if  $M = 2^k$  for some  $k$ ,  $\operatorname{Tr} A \leq c^2$ . In general,  $\operatorname{Tr} A \leq (\log_2 M) c^2$ .

Proof. We use the fact, that if  $y^1, y^2, \dots, y^M$  are any orthonormal set of  $M$  dimensional (row) vectors, then  $\operatorname{Tr} A = \sum_{k=1}^M y^k A y^{k'}$ . If  $M = 2^k$ , for some integer  $k$ , then there exists a set of  $M$  orthonormal vectors, each of the form  $y^k = \frac{1}{\sqrt{M}} (y_1^k, y_2^k, \dots, y_M^k)$  where  $y_{\ell}^k, \ell = 1, 2, \dots, M$  is  $\pm 1$ . In this case, the hypothesis gives  $y^k A y^{k'} \leq \frac{c^2}{M}$  and  $\operatorname{Tr} A \leq c^2$ . In general, if  $2^k < M < 2^{k+1}$ , write  $M = \sum_{v=0}^k \theta_v 2^v$  where  $\theta_k = 1$  and  $\theta_v = 0$  or  $1$ ,  $v = 0, 1, 2, \dots, k-1$ . Let  $\ell$  be the number of non-zero  $\theta_v$ 's. An  $M \times M$  orthogonal matrix can be constructed with  $\ell$  non-zero blocks down the diagonal, the  $n^{\text{th}}$  block of dimension  $2^{v_m} \times 2^{v_m}$ ,  $m = 1, 2, \dots, \ell$ , where  $v_m$  corresponds to the  $m^{\text{th}}$  non-zero  $\theta_v$ . In the  $m^{\text{th}}$  block place an orthogonal matrix with rows of the form  $\frac{1}{\sqrt{2^{v_m}}} (u_1, u_2, \dots, u_{2^{v_m}})$  where  $u_r = \pm 1$ . Now let  $y^1, y^2, \dots, y^M$  be the rows of this matrix. If part of  $y^k$  is contained in the  $m^{\text{th}}$  block,  $y^k A y^{k'} \leq \frac{c^2}{2^{v_m}}$ , and there are  $2^{v_m}$  such  $y^k$ 's. Hence

$$\operatorname{Tr} A \leq \sum_{m=1}^{\ell} 2^{v_m} \cdot \frac{c^2}{2^{v_m}} = \ell c^2 \leq k c^2 \leq (\log_2 M) c^2 .$$

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