DEPARTMENT OF STATISTICS

The University of Wisconsin
Madison, Wisconsin

TECHNICAL REPORT NO. 187

December 1968

SOME TESTS OF INDEPENDENCE FOR STATIONARY MULTIVARIATE TIME SERIES

by

Grace Wahba

University of Wisconsin*

The research was supported in part by the National Science Foundation under Grant GP-4265 while the author was at Stanford University, and the Wisconsin Alumni Research Foundation.

Abstract

Let

$$X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ \cdot \\ \cdot \\ \cdot \\ X_p(t) \end{pmatrix} \quad , t = \ldots -1, 0, 1, \ldots$$

be a P-dimensional zero-mean stationary Gaussian time series possesing a spectral density matrix

$$F(\omega) = \{f_{ij}(\omega)\}_{i,j=1}^{P}$$

satisfying some mild regularity conditions. Let $\hat{F}(\omega_\ell)$, $\ell = 1, 2, \ldots M$, $\hat{F}(\omega_\ell)=\{\hat{f}_{ij}(\omega_\ell)\}$ be suitably defined sample spectral density matrices for M values of ω, based on a record of length $T >> M$. We consider the following (null) hypotheses

H_1: $X_i(t)$, $X_j(s)$ independent if $i \neq j$, all s, t

H_2: $X_1(t)$ independent of $X_j(s)$, $j = 2, \ldots P$, all s, t

H_3: $X_1(t)$, $X_1(t+\tau)$ independent all $\tau \neq 0$

Approximate likelihood ratio tests are derived and the test statistics λ_i, i = 1, 2, 3 are found to be functions of the

$\{\hat{F}(\omega_\ell)\}_{\ell=1}^{M}$. The $\{\hat{F}(\omega_\ell)\}$ are shown to converge in mean square to a family of independent complex Wishart matrices. Using this fact, $\log \lambda_1$ and $\log \lambda_2$ are shown to converge in first mean to random variables whose null densities can be given explicitly, being distributed as the logs of products of independent beta random variables with integer indices. Under the null hypothesis λ_3 is distributed exactly as $\underset{\sim}{\lambda}_3$ where $\underset{\sim}{\lambda}_3$ is Bartletts statistic for homogeneity of variances. The distributions of $\log \lambda_i$, $i = 1, 2, 3$ under the alternative are discussed. Under the alternative, $\log \lambda_3$ tends in first mean to $\log \underset{\sim}{\lambda}_3$. A power series expansion for the characteristic function of $\log \underset{\sim}{\lambda}_3$ under the alternative is given.

SOME TESTS OF INDEPENDENCE

FOR STATIONARY MULTIVARIATE TIME SERIES

By

Grace Wahba

University of Wisconsin

1. Introduction

Let

$$X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ \cdots \\ X_P(t) \end{pmatrix}, \quad t = \ldots -1, 0, 1, \ldots$$

be a P-dimensional zero mean stationary Gaussian time series, possesing a strictly positive definite spectral density matrix $F(\omega)$ satisfying regularity conditions to be stated. The observed data consists of a single record of length T.

In this note we consider asymptotic likelihood ratio tests for the following three null hypotheses against the above general alternative.

H_1: $X_i(t)$, $X_j(s)$ independent if $i \neq j$, all s, t

H_2: $X_1(t)$ independent of $X_j(s)$, $j = 2, \ldots, P$, all s, t

H_3: $X_1(t)$, $X_1(t+\tau)$ independent all $\tau \neq 0$ ("white noise").

These hypotheses are equivalently stated as

H_1: $F(\omega)$ diagonal, all ω

H_2: $f_{1j}(\omega) = 0$, $j = 2, \ldots, P$, all ω

H_3: $f_{11}(\omega) =$ constant .

where

$$F(\omega) = \{f_{ij}(\omega)\}_{P\times P} \tag{1.1}$$

$$f_{ij}(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} e^{-i\omega\tau} R_{ij}(\tau) \tag{1.2}$$

$$R_{ij}(\tau) = E[X_i(t)\, X_j(t+\tau)] \tag{1.3}$$

First observe that, in the case of H_1 and H_2, values of PT random variables are observed, where as the hypotheses involve a countably infinite number of parameters. In fact, the number of parameters involved in the joint density of $X(t)$, $t = 1, 2, \ldots T$ is $PT + (2T-1)\, P(P-1)/2$. To have the problem make sense, then we must assume that some of the parameters are negligible. In this paper we always assume

Condition A $$\sum_{\tau=-\infty}^{\infty} \sum_{i,j=1}^{P} |R_{ij}(\tau)|\,|\tau| = \theta < \infty \tag{1.4}$$

This condition insures that the entries $f_{ij}(\omega)$ of the spectral density matrix exist and have derivatives bounded by θ, and that

$$\Lambda(\omega) \leq \Lambda/2\pi < \infty \tag{1.5a}$$

where $\Lambda(\omega)$ is the largest eigenvalue of $F(\omega)$. We also assume

Condition B $$\lambda(\omega) \geq \lambda/2\pi > 0 \tag{1.5b}$$

where $\lambda(\omega)$ is the smallest eigenvalue of $F(\omega)$. Condition B guarantees that the process is non-degenerate.

Utilizing the assumption of boundedness of the derivatives of the entries, we will approximate $F(\omega)$ by a matrix of step functions $\bar{F}(\omega)$ which involves a reduced number of parameters, and then consider hypothesis tests involving $\bar{F}(\omega)$. This is done as follows. Choose n, M and T integers satisfying $(2n+1)M = (T-1)/2$ (T is odd, without appreciable loss of generality). Let $j_\ell = (\ell-1)(2n+1) + (n+1)$, $\ell = 1, 2, \ldots M$, let $\omega_\ell = 2\pi j_\ell/T$ and define $\bar{F}_{n,M,T}(\omega_\ell) = \bar{F}(\omega_\ell)$ as

$$\bar{F}(\omega_\ell) = \frac{1}{2n+1}\sum_{j=-n}^{n} F(\omega_\ell+2\pi j/T) \quad , \quad \ell = 1, 2, \ldots M \tag{1.6}$$

Extend $\bar{F}(\omega_\ell)$ to a matrix $\bar{F}(\omega)$ of step functions on $0 \le \omega \le 2\pi$ by letting $\bar{F}$ be continuous from the left, say on $0 < \omega \le \pi$ and defined by

$$\begin{aligned} \bar{F}(\omega) &= \bar{F}(\omega_\ell) \quad \text{all } \omega \ni |\omega-\omega_\ell| < \min_{j\neq\ell} |\omega-\omega_j|, \ 0 < \omega < \pi \\ \bar{F}(\omega) &= \bar{F}^*(2\pi-\omega) \qquad \pi < \omega < 2\pi \\ \bar{F}(0) &= \bar{F}(2\pi) = F(0) \end{aligned} \tag{1.7}$$

Thus $\bar{F}(\omega)$ is a step function approximation to $F(\omega)$ where the "steps", except at the end, are of width $2\pi(2n+1/T) \approx \pi/M$.

$\bar{F}(\omega)$ provides an approximation $F(\omega)$ which is everywhere good to an accuracy of at least $2\theta(2n+1)/T$ and hence $|\bar{F}(\omega)-F(\omega)|$ tends uniformly to 0 as $M \to \infty$. Next, let X be the $P \times T$ matrix of random variables with i, s-th entry $X_i(s)$, $i = 1, 2, \ldots P$, $s = 1, 2, \ldots T$. For each n, M, we will define, on the same sample space as X, a $P \times T$ matrix of normal, zero mean random variables $\underset{\sim}{X} = \{\underset{\sim}{X}_i(s)\ i = 1, 2, \ldots P, s = 1, 2, \ldots T\}$ which will approximate X in a suitable sense, and whose joint density depends only on the M matrices $\bar{F}(\omega_\ell)$, $\ell = 1, 2, \ldots M$. Consider the "approximate" hypotheses

$$\begin{aligned} &\underset{\sim}{H}_1: \bar{F}(\omega) \text{ diagonal } 0 < \omega < 2\pi \quad [1] \\ &\underset{\sim}{H}_2: \bar{f}_{1j}(\omega) = 0,\ j = 2, \ldots P,\ 0 < \omega < 2\pi \\ &\underset{\sim}{H}_3: \bar{f}_{11}(\omega) \text{ constant } 0 < \omega < 2\pi. \end{aligned} \tag{1.8}$$

Where we have written $\bar{F}(\omega) = \{\bar{f}_{ij}(\omega),\ i, j = 1, 2, \ldots P\}$. We first find the likelihood ratio statistics $\lambda_i(\underset{\sim}{X})$, $i = 1, 2, 3$, for $\underset{\sim}{H}_i$, based on $\underset{\sim}{X}$. These statistics have well known analogues in ordinary multivariate analysis. We discuss their distributions in Section 7. The random matrix $\underset{\sim}{X}$ is not observable, however, since its construction involves unknown parameters. Let $\lambda_i(X)$ be the same statistic calculated from the observations X.

[1] To avoid uninteresting difficulties which are negligible we omit $\omega = 0$ here.

The major result, which renders $\lambda_i(X)$ useful is the following theorem.

Theorem 2. Suppose conditions A and B be satisfied, and suppose n, $M \to \infty$ in such a way that $(\log M)/n \to 0$. Let $c_i^{-1} = c_i^{-1}(n,M)$ be the standard deviation of $\log \lambda_i(\underset{\sim}{X})$. Then

$$E\, c_i \left|\log \lambda_i(X) - \log \lambda_i(\underset{\sim}{X})\right| \to 0,\ i = 1,\ 2,\ 3. \tag{1.9}$$

For $Y = \{Y_i(s),\ i = 1, 2, \ldots P,\ s = 1, 2, \ldots T\}$ a $P \times T$ matrix of random variables, define the sample spectral density matrices $\hat{F}(\omega_\ell, n, M, Y) = \hat{F}(\omega_\ell, Y)$ as

$$\hat{F}(\omega_\ell, Y) = \{\hat{f}_{\mu\nu}(\omega_\ell, Y)\}_{\mu,\nu=1}^{P}$$

(1.10)

$$\hat{f}_{\mu,\nu}(\omega_\ell, Y) = \frac{1}{(2n+1)} \sum_{j=-n}^{n} \frac{1}{2\pi T} \sum_{s,t=1}^{T} Y_\mu(s) e^{is(\omega_\ell + 2\pi j/T)} Y_\nu(t) e^{-it(\omega_\ell + 2\pi j/T)}$$

$$\ell = 1,\ 2,\ \ldots\ M.$$

The sample spectral density matrices so defined correspond to taking averages of 2n+1 neighboring periodograms. The random matrix $\underset{\sim}{X}$ is defined to have the property that

$$\{\hat{F}(\omega_\ell, \underset{\sim}{X})\},\quad \ell = 1,\ 2,\ \ldots\ M$$

are a set of independent Complex Wishart distributed random matrices. The Complex Wishart distribution, first introduced by Goodman [4], has properties analoguous to the usual (real) Wishart distribution and is described in Section 4. To obtain Theorem 2, we use Theorem 1, of independent interest.

Theorem 1. Under conditions A and B,

$$E\ T_r \sum_{\ell=1}^{M} [(\hat{F}(\omega_\ell, X) - \hat{F}(\omega_\ell, \underset{\sim}{X})][(\hat{F}(\omega_\ell, X) - \hat{F}(\omega_\ell \underset{\sim}{X})]^{*'}$$

$$= 0[(\log M)/n^2] \tag{1.11}$$

Using the properties of the Complex Wishart distribution, it will follow that the likelihood ratio statistics $\lambda_i(\underset{\sim}{X}) = \lambda_i(\underset{\sim}{X}, n, M)$ for $\underset{\sim}{H}_i$, i = 1, 2, 3, are given by

$$\underset{\sim}{H}_1:\ \lambda_1(\underset{\sim}{X}) = \prod_{\ell=1}^{M} \left(\frac{|\hat{F}(\omega_\ell, \underset{\sim}{X})|}{\prod_{j=1}^{P} \hat{f}_{jj}(\omega_\ell, \underset{\sim}{X})} \right) \tag{1.12a}$$

$$\underset{\sim}{H}_2:\ \lambda_2(\underset{\sim}{X}) = \prod_{\ell=1}^{M} \left(\frac{|\hat{F}(\omega_\ell, \underset{\sim}{X})|}{\hat{f}_{11}(\omega_\ell, \underset{\sim}{X})\ |\hat{F}_{22}(\omega_\ell, \underset{\sim}{X})|} \right),\quad \hat{F}_{22} = \{\hat{f}_{ij}\}_{i,j=2}^{P} \tag{1.12b}$$

$$\underset{\sim}{H}_3: \lambda_3(\underset{\sim}{X}) = \frac{\left(\prod_{\ell=1}^{M} \hat{f}_{11}(\omega_\ell, \underset{\sim}{X})\right)}{\left(\frac{1}{M}\sum_{\ell=1}^{M} \hat{f}_{11}(\omega_\ell, \underset{\sim}{X})\right)^M} \tag{1.12c}$$

Each term in the product in $\lambda_1(\underset{\sim}{X})$ and $\lambda_2(\underset{\sim}{X})$ has a well known real multivariate analogue, as follows. Suppose $x = (x_1, x_2, \ldots x_P)$ is a normal random vector with mean 0 and covariance matrix $S_{(P\times P)}$, and let $\hat{S} = \{\hat{s}_{ij}\}$ be the sample co-variance matrix based on n in dependent observations on x. Then the likelihood ratio statistic ν_1 for A_1: S diagonal is

$$\nu_1 = \frac{|\hat{S}|}{\prod_{j=1}^{P} \hat{s}_{jj}} \tag{1.13.a}$$

and the likelihood ration statistic ν_2 for A_2: $s_{ij} = 0$ $j = 2, 3, \ldots P$, is

$$\nu_2 = \frac{|\hat{S}|}{\hat{s}_{11}|\hat{S}_{22}|}, \quad \hat{S} = \begin{pmatrix} \hat{s}_{11} & \hat{s}_{12} \\ \hat{s}_{12}' & \hat{S}_{22} \end{pmatrix} \tag{1.13.b}$$

(See e.g. Anderson [1]). This similarity is not surprising, since $F(\omega)$ and S share a number of theoretical properties [see [7]]. Under A_1 and A_2, these statistics are well known to be distributed as products of independent Beta random variables, with, in general, non-integer indices [1]; numerous authors [for example [3], [8]) have discussed methods for approximating the densities. In going from the real to the complex case, we generally find that the number of degrees of freedom is doubled, here, each term in the product in (1.12a) and (1.12b) may be shown to be distributed as the product of indpendent Beta random variables with integer indices. A simple procedure is used to exhibit the exact null moments and densities of $\log \lambda_1(\underset{\sim}{X})$ and $\log \lambda_2(\underset{\sim}{X})$, and the asymtotic means and variances (large n) of $\log \lambda_1(\underset{\sim}{X})$ and $\log \lambda_2(\underset{\sim}{X})$ under the alternative are given.

It is shown that $\lambda_3(\underset{\sim}{X}) = \underset{\sim}{\lambda}_3$ is distributed as Bartlett's statistic for homogeneity of variances, with the appropriate choices of degrees of freedom. This statistic has been discussed by numerous authors. Under the null hypothesis $\underset{\sim}{\lambda}_3$ is distributed (in general) as a product of independent Beta random variables with non-integer indices. Wilks [11] gave the moments of $\underset{\sim}{\lambda}_3^{-1}$ under the alternative, Whittle [10] discussed $\underset{\sim}{\lambda}_3$ in the context of $\underset{\sim}{H}_3$ and gave the null characteristic function and cumulants for $\log \underset{\sim}{\lambda}_3$ with $n = 1$. Very little seems to be known about the form of the alternative density. The characteristic function of $\log \underset{\sim}{\lambda}_3$ under the alternative is here expressed as an infinite weighted sum of characteristic functions. A nearby alternative may be defined as one for which $\text{var}\, f_{11}(\omega)/[\bar{f}_{11}(\omega)]^2$ is small, where

$$\text{var}\, f_{11}(\omega) = \frac{1}{2\pi}\int_0^{2\pi} (f_{11}(\omega) - \bar{f}_{11})^2\, d\omega$$

$$\bar{f}_{11} = \frac{1}{2\pi}\int_0^{2\pi} f(\omega)\, d\omega$$

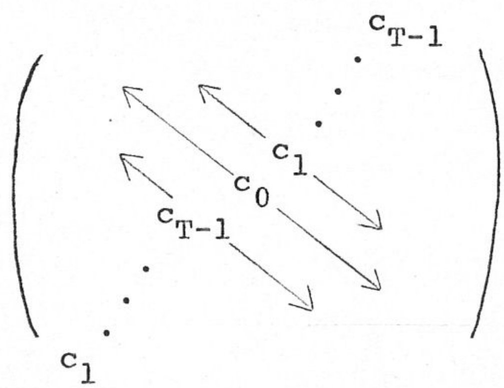
For a nearby alternative the first few terms in the expression for the characteristic function of $\log \underset{\sim}{\lambda}_3$ should suffice. An approximate expression for the first two moments of $\log \underset{\sim}{\lambda}_3$ for nearby alternatives is given.

2. Circulant Matrices

In this section we give two lemmas about circulant matrices which will be used in the sequel. Lemma 2.1, which is well known, shows that all circulant matrices commute, and exhibits the eigenvalues in terms of the elements. Lemma 2.2 will be used in approximating a Toeplitz matrix by a circulant matrix.

<u>Lemma 2.1.</u> Let C (circulant, real or complex) be of the form

C =



Let $K(\omega) = \frac{1}{2\pi} \sum_{\tau=0}^{T-1} c_\tau e^{-i\omega\tau}$, let W be the $T \times T$ unitary matrix with r, s-th element $\frac{1}{\sqrt{T}} e^{2\pi i r s/T}$ and let D_K be the $T \times T$ diagonal matrix with r, r-th element $K(\frac{2\pi r}{T})$. Then

$$C = 2\pi W D_K W^* .$$

Proof.

$$[2\pi W D_K W^*]_{r,s} = \frac{2\pi}{T} \sum_{\nu=1}^{T} e^{2\pi i (\frac{r-s}{T})\nu} K(\frac{2\pi\nu}{T})$$

$$= \frac{1}{T} \sum_{\nu=1}^{T} \sum_{\tau=0}^{T-1} e^{2\pi i (\frac{r-s}{T})\nu} e^{-\frac{2\pi i \nu\tau}{T}} c_\tau$$

$$= \frac{1}{T} \sum_{\tau=0}^{T-1} c_\tau \sum_{\nu=1}^{T} e^{2\pi i (\frac{r-s-\tau}{T})\nu} .$$

Since

$$\frac{1}{T} \sum_{\nu=1}^{T} e^{\frac{2\pi i \nu}{T}(r-s-\tau)} = 1 \quad \tau = (r-s) + \ell T, \quad \ell = 0, \pm 1, \pm 2, \ldots,$$

$$= 0 \quad \text{otherwise},$$

we have

$$[2\ W D_K W^*]_{r,s} = c_{r-s}, \qquad r - s \geq 0,$$

$$= c_{T-|(r-s)|}, \quad r - s < 0.$$

Remarks. C Hermitian => $K(\frac{2\pi\nu}{T})$ is real for all integers ν, C real => $K(\frac{2\pi\nu}{T}) = K^*(\frac{2\pi(T-\nu)}{T})$.

Lemma 2.2 Let $R(\tau)$, $\tau = \ldots -1, 0, 1,$ be a doubly infinite sequence of real numbers with

$$\sum_{\tau=-\infty}^{\infty} |\tau|\ |R(\tau)| = \theta < \infty,$$

let

$$f(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} R(\tau) e^{-i\omega\tau}$$

and for fixed T, let D_f be the T × T diagonal matrix with r, r-th entry $f(\frac{2\pi r}{T})$. Let W be as in Lemma 1, and let

$$\tilde{\Sigma} = \{\tilde{\sigma}_{\mu\nu}, \mu, \nu = 1, 2, \ldots, T\} = 2\pi W D_f W^*$$

$$\Sigma = \{\sigma_{\mu\nu}, \mu, \nu = 1, 2, \ldots, T\} \quad \sigma_{\mu\nu} = R(\mu-\nu).$$

Note that $\tilde{\Sigma}$ and Σ, are (general) circulant and Toeplitz matrices, respectively. For any matrices of the same dimensions, define

$$\phi(A-B) = \sum_{\mu,\nu} |a_{\mu\nu}-b_{\mu\nu}|, \quad A = \{a_{\mu\nu}\}, \quad B = \{b_{\mu\nu}\}.$$

Then

$$\phi(\Sigma-\tilde{\Sigma}) = \sum_{\mu,\nu=1}^{T} |\sigma_{\mu\nu}-\tilde{\sigma}_{\mu\nu}| \leq 2\theta.$$

<u>Proof</u>. By a calculation similar to that of Lemma 2.1,

$$\tilde{\sigma}_{\mu\nu} = \sum_{\ell=-\infty}^{\infty} R(\mu-\nu+\ell T)$$

$$\sum_{\mu,\nu=1}^{T} |\sigma_{\mu\nu}-\tilde{\sigma}_{\mu\nu}| = \sum_{\tau=-(T-1)}^{T-1} (T-|\tau|) \left| \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} R(\tau+\ell T) \right|$$

$$\leq \sum_{\tau=-(T-1)}^{T-1} (T-|\tau|) \sum_{\substack{\ell=-\infty \\ \ell\neq 0}}^{\infty} |R(\tau+\ell T)|$$

$$\leq T \left\{ \sum_{\substack{\ell=-\infty \\ \ell\neq 0,-1,+1}}^{\infty} \sum_{\tau=-(T-1)}^{T-1} |R(\tau+\ell T)| + \sum_{\tau=0}^{T-1} |R(\tau+T)| + \sum_{\tau=-(T-1)}^{0} |R(\tau-T)| \right\}$$

$$+ \sum_{\tau=-(T-1)}^{-1} (T-|\tau|) |R(\tau+T)| + \sum_{\tau=1}^{T-1} (T-|\tau|) R(\tau-T)$$

$$\leq 2T \sum_{|\tau|\geq T} |R(\tau)| + \sum_{\tau=-(T-1)}^{T-1} |\tau||R(\tau)| \leq 2 \sum_{-\infty}^{\infty} |\tau| |R(\tau)| = 2\theta$$

3. Two random matrices approximating X

Let X_i, i = 1, 2, ... P be the i-th row of X, that is

$$X_i = (X_i(1), X_i(2), \dots X_i(T))$$

Define the T × T matrices $\sum_{ij}$, i, j = 1, 2, ... P by

$$E\, X_i'\, X_j = \sum_{ij} \qquad (3.1)$$

($\sum_{ij}$ has μ,νth entry $R_{ij}(\mu-\nu)$.) Let $\sum$ be the PT × PT block Toeplitz matrix with P×P blocks of dimension T×T, with $\sum_{ij}$ in the ij-th block.

Let $\tilde{D}_{ij}$ be the T × T diagonal matrix with r r-th element $f_{ij}(2\pi r/T)$, i, j = 1, 2, ... P and define the T × T circulant matrix $\tilde{\sum}_{ij}$ by

$$\tilde{\sum}_{ij} = 2\pi\ W\tilde{D}_{ij}W^* \qquad (3.2)$$

Let $\tilde{\sum}$ be the PT × PT matrix of P × P blocks of dimension T × T with $\tilde{\sum}_{ij}$ in the ij-th block. Let $\tilde{X}_i = (\tilde{X}_i(1), \tilde{X}_i(2), \ldots \tilde{X}_i(T))$, i = 1, 2, ... P be the PT random variables defined by

$$(\tilde{X}_1, \tilde{X}_2, \ldots \tilde{X}_P) = (X_1, X_2, \ldots X_p)\ \sum^{-1/2}\ \tilde{\sum}^{1/2} \qquad (3.3)$$

where, for A real (or Hermitian) strictly positive definite, $A^{1/2}$ is the symmetric (or Hermitian) square root. [Condition B insures that $\sum$ is invertible, see [5]. We have that $E\tilde{X}_i'\tilde{X}_j = \tilde{\sum}_{ij}$. The covariance matrix $\tilde{\sum}$ of $(\tilde{X}_1, \tilde{X}_2, \ldots \tilde{X}_P)$ is "close" to the covariance matrix $\sum$ of $(X_1, X_2, \ldots X_P)$, since condition A and Lemma 2.2 insure that

$$\phi(\sum - \tilde{\sum}) \leq 2\theta \qquad (3.4)$$

independently of T, where $\phi(\cdot)$ is defined in section 2 and θ is given by (1.4). The random matrix X whose is-th entry is $X_i(s)$, i = 1, 2, ... P, s = 1, 2, ... T, will be used in the proof of Theorem 1.

Let $\underset{\sim}{D}_{ij}$ be the T × T diagonal matrix with r r-th element $\bar{f}_{ij}(\frac{2\pi r}{T})$, i, j = 1, 2, ... P, where $\bar{f}_{ij}(\omega) = \bar{f}_{ij}(\omega,n,M,T)$ is defined by (1.7), $\{\bar{f}_{ij}(\omega)\} = \bar{F}(\omega)$. Let $\underset{\sim}{\Sigma}_{ij}$ be defined by

$$\underset{\sim}{\Sigma}_{ij} = 2\pi \; W\underset{\sim}{D}_{ij}W^*, \tag{3.5}$$

and let $\underset{\sim}{\Sigma}$ be the PT × PT matrix of P × P blocks of dimension T × T with $\underset{\sim}{\Sigma}_{ij}$ in the i, j-th block. Let $\underset{\sim}{X}_i = (\underset{\sim}{X}_i(1), \underset{\sim}{X}_i(2), \ldots \underset{\sim}{X}_i(T))$, i = 1, 2, ... P, be the PT random variables defined by

$$(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots \underset{\sim}{X}_P) = (\tilde{X}_1, \tilde{X}_2, \ldots \tilde{X}_P)\; \tilde{\Sigma}^{-1/2} \; \underset{\sim}{\Sigma}^{1/2} \tag{3.6}$$

Hence the covariance matrix of $(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots \underset{\sim}{X}_P)$ is $\underset{\sim}{\Sigma}$. Loosely speaking, $\underset{\sim}{\Sigma}$ is "close" to $\tilde{\Sigma}$, since $\bar{F}(\omega)$ is "close" to $F(\omega)$. Letting $\underset{\sim}{X}$ be the P × T random matrix whose i-th row is $\underset{\sim}{X}_i$, we have the joint density of $\underset{\sim}{X}$ depends only on the parameters $\bar{F}(\omega_\ell)$, ℓ = 1, 2, ... M, and $\bar{F}(0)$. We will derive the likelihood ratio statistics for $\underset{\sim}{H}_i$, based on $\underset{\sim}{X}$. To do this we need the notions of complex Normal and complex Wishart distributions, which are described in the next section. For later reference, note that

$$(\underset{\sim}{X}_1, \underset{\sim}{X}_2, \dots \underset{\sim}{X}_P) = \underset{\sim}{\Sigma}^{1/2} \Sigma^{-1/2} (X_1, X_2, \dots X_P) \qquad (3.7)$$

4. Likelihood ratio statistics for $\underset{\sim}{H}_i$, based on $\underset{\sim}{X}$.

Restating ideas introduced by Goodman [4] we first describe the complex Normal random vector and the complex Wishart random matrix. $Z = (Z_1, Z_2, \dots Z_P)' = U + iV$ is said to be a P dimensional complex normal random vector (with zero mean) if $U = (U_1, U_2, \dots U_P)'$ and $V = (V_1, V_2, \dots V_P)'$ are two real P dimensional normal random vectors with the following special covariance structure

$$\begin{pmatrix} EU_jV_k & EU_jV_k \\ EU_kV_j & EV_jV_k \end{pmatrix} \begin{matrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} f_{kk} \quad \text{if } j = k = 1, 2, \dots P \\ = \frac{1}{2} \begin{bmatrix} c_{jk} & -q_{jk} \\ q_{jk} & c_{jk} \end{bmatrix} \quad j \neq k \end{matrix} \qquad (4.1)$$

It is shown in [4] that

$$E\, Z\, Z^{*'} = F \qquad (4.2)$$

where F is the hermitian matrix with jk-th entry f_{kk}, $j = k$, and $f_{jk} = c_{jk} + iq_{jk}$, $j < k$, $(q_{jk} = -q_{kj})$, and that the density function for the 2 P random variables Z may be written

$$P(Z) = \frac{1}{(\pi)^P} \frac{1}{|F|} e^{-Z' F^{-1} Z} . \tag{4.3}$$

Given 2n+1 independent samples Z_r, r = 1, 2, ... 2n+1, from the density P(Z), the maximum likelihood estimate $\hat{F}$ for F is

$$\hat{F} = \frac{1}{2n+1} \sum_{r=1}^{n} Z_r Z_r^{*'} . \tag{4.4}$$

$\hat{F}$ is a sufficient statistic for F and has the complex Wishart distribution, denoted W_c(F, P, 2n+1), with density

$$P(\hat{F}) = (\pi^{\frac{1}{2}P(P-1)} \Gamma(P+2n+1) \ldots \Gamma(1+2n+1) |F|^{P+2n+1})^{-1}$$

$$|\hat{F}|^{2n+1} e^{-\mathrm{tr}\, F^{-1}\hat{F}} \tag{4.5}$$

The joint density of $\underset{\sim}{X}$ is given by

$$P(\underset{\sim}{X}) = (2\pi^{PT/2} |\underset{\sim}{\Sigma}|^{1/2})^{-1} e^{-\frac{1}{2} \sum_{i,j} \underset{\sim}{X}_i \underset{\sim}{\Sigma}^{ij} \underset{\sim}{X}_j'} \tag{4.6}$$

where $\underset{\sim}{\Sigma}^{ij}$ is the (i,j)th block in $\underset{\sim}{\Sigma}^{-1}$. Let $\underset{\sim}{Z}_j$ be the complex random vector defined by

$$\underset{\sim}{Z}_j = (\underset{\sim}{Z}_j(1), \underset{\sim}{Z}_j(2), \ldots \underset{\sim}{Z}_j(T) = \underset{\sim}{X}_j W, \tag{4.7}$$

we have $\underset{\sim}{Z}_j(r) = \underset{\sim}{Z}_j^*(T-r)$. $\underset{\sim}{Z}_j(T)$ and, if T is even, $\underset{\sim}{Z}_{ij}(T/2)$ are real. For T odd the transformation (4.7) between the T components of $\underset{\sim}{X}_j$ and the (T-1)/2 real parts of $\underset{\sim}{Z}_j(r)$, r = 1, ... (T-1)/2, the (T-1)/2 complex parts of $\underset{\sim}{Z}_j(r)$, $Z_j(T)$ is orthogonal. Let $\underset{\sim}{Z}(r)$ be the P-dimensional (column) vector given by the r-th column of $\underset{\sim}{X}\,W$, that is,

$$Z(r) = \begin{pmatrix} Z_1(r) \\ Z_2(r) \\ \cdot \\ \cdot \\ \cdot \\ Z_P(r) \end{pmatrix} \tag{4.8}$$

It follows (T odd) from (3.5) (4.6), (4.7) and (4.8) and the fact (see [4]) that $|2\sum|^{1/2} = |2\pi\bar{F}(2\pi)|^{1/2} \prod_{r=1}^{(T-1)/2} |2\pi\bar{F}(2\pi r/T)|$, that the joint density of $\underset{\sim}{Z}(r)$, r = 1, 2, ... (T-1)/2, T, is given by

$$P(\underset{\sim}{Z}(r), r=1,2,\ldots(T-1)/2,T) = \left(\pi^{P(T-1)/2} \prod_{r=1}^{(T-1)/2} |2\pi\bar{F}(2\pi r/T)|\right)^{-1} \times$$

$$e^{-\sum_{r=1}^{(T-1)/2} \underset{\sim}{Z}'(r)(2\pi\bar{F}(2\pi r/T))^{-1}\underset{\sim}{Z}^*(r)} \times$$

$$((2\pi)^P|2\pi\bar{F}(2\pi)|)^{-1/2}\, e^{-\frac{1}{2}\underset{\sim}{Z}'(T)(2\pi\bar{F}(2\pi))^{-1}\underset{\sim}{Z}^*(T)} \quad [2] \tag{4.9}$$

[2] The apparently artificial definition of $\bar{F}(2\pi) = F(2\pi)$ is to guarantee that $\bar{F}(2\pi)$ is real here. The statements of $\underset{\sim}{H}_i$ excluding $F(2\pi)$ are to avoid laborious but uninteresting complications due to the fact that $\underset{\sim}{Z}(T)$ is real.

We may rewrite (4.9) slightly, using the definition of $F(\omega)$, as

$$P(\underset{\sim}{Z}(r),\ r=1,2,\dots (T-1)/2,T) = \prod_{\ell=1}^{M} (\pi^{P}|2\pi\bar{F}(\omega_\ell)|)^{-(2n+1)} \times e^{-\sum_{\nu=-n}^{n} \underset{\sim}{Z}'(j_\ell+\nu)(2\pi\bar{F}(\omega_\ell))^{-1}\underset{\sim}{Z}^{*}(j_\ell+\nu)} \times (2\pi)^{P}|2\pi\bar{F}(2\pi)|^{-1/2} e^{-\frac{1}{2}\underset{\sim}{Z}'(T)(2\pi\bar{F}(2\pi))^{-1}Z^{*}(T)}$$

It follows from the remarks at the beginning of this section concerning complex normal vectors, that sufficient statistics for $\bar{F}(\omega_\ell)$, $\ell = 1, 2, \dots M$ are the so-called sample complex covariance matrices,

$$\frac{1}{2\pi}\left(\frac{1}{2n+1}\right)\sum_{\nu=-n}^{n} \underset{\sim}{Z}(j_\ell+\nu)\ \underset{\sim}{Z}^{*'}(j_\ell+\nu), \qquad \ell = 1, 2, \dots M$$

By observing that the ν-th component of $\underset{\sim}{Z}(j_\ell+\nu)$ is

$$\frac{1}{\sqrt{T}}\sum_{s=1}^{T} \underset{\sim}{X}_\nu(s)\ e^{2\pi i(j_\ell+\nu)s}$$

we have

$$\frac{1}{2\pi(2n+1)}\sum_{\nu=-n}^{n} \underset{\sim}{Z}(j_\ell+\nu)\ \underset{\sim}{Z}^{*'}(j_\ell+\nu) = \hat{F}(\omega_\ell,\underset{\sim}{X}) \tag{4.11}$$

where $\hat{F}(\omega_\ell,Y)$ is defined by 1.10.

Furthermore, the $\{\hat{F}(\omega_\ell, X),\}_{\ell=1}^{M}$ are M independent complex Wishart matrices, $W_c(\bar{F}(\omega_\ell), P, 2n+1)$. Following the differentiation argument of Khatri [5], Section 3, the likelihood ratio statistics for $\underset{\sim}{H}_1$, $\underset{\sim}{H}_2$ and $\underset{\sim}{H}_3$ based on the random variables $\underset{\sim}{X}$ are

$$\underset{\sim}{H}_1: \lambda_1(n,M,\underset{\sim}{X}) = \lambda_1(\underset{\sim}{X}) = \prod_{\ell=1}^{M} \left(\frac{|\hat{F}(\omega_\ell, \underset{\sim}{X})|}{\prod_{j=1}^{P} \hat{f}_{jj}(\omega_\ell, \underset{\sim}{X})} \right), \tag{4.12}$$

$$\hat{F}(\omega_\ell, \underset{\sim}{X}) = \{\hat{f}_{ij}(\omega_\ell, \underset{\sim}{X}),\ i,\ j=1,2,\ldots P\}$$

$$\underset{\sim}{H}_2: \lambda_2(n,M,\underset{\sim}{X}) = \lambda_2(\underset{\sim}{X}) = \prod_{\ell=1}^{M} \left(\frac{|\hat{F}(\omega_\ell, \underset{\sim}{X})|}{\hat{f}_{11}(\omega_\ell, \underset{\sim}{X})\,|\hat{F}_{22}(\omega_\ell, \underset{\sim}{X})|} \right) \tag{4.13}$$

$$\hat{F}_{22}(\omega_\ell, \underset{\sim}{X}) = \{\hat{f}_{ij}(\omega_\ell, \underset{\sim}{X}),\ i,\ j=2,3,\ldots P\}$$

$$H_3: \lambda_3(n,M,\underset{\sim}{X}) = \lambda_3(\underset{\sim}{X}) = \frac{\left(\prod_{\ell=1}^{M} \hat{f}_{11}(\omega_\ell, \underset{\sim}{X}) \right)}{\left(\frac{1}{M} \sum_{=1}^{M} \hat{f}_{11}(\omega_\ell, \underset{\sim}{X}) \right)^{M}} \tag{4.14}$$

The random variables $\underset{\sim}{X}$ are not observable, since the transformation from X to $\underset{\sim}{X}$ defined by (3.7) is unknown. Hence $\lambda_i(\underset{\sim}{X})$ is not computable. Let $\lambda_i(X)$ be the (computable) statistics defined by (4.12), (4.13) and (4.14) with $\underset{\sim}{X}$ replaced by X.

Towards our goal of proving Theorem 2:

$$E\, c_i \,|\log \lambda_i(X) - \log \lambda_i(\underset{\sim}{X})| \to 0,$$

in the next section we prove that $\{\hat{F}(\omega_\ell,X) \to \hat{F}(\omega_\ell,\underset{\sim}{X}),\ \ell = 1,2,\ldots M\}$ in a suitable sense.

5. $\{\hat{F}(\omega_\ell,X)\}_{\ell=1}^{M}$ are asymptotically complex Wishart.

The arguments of this paper rest on Theorem 1, below, which is proved in this section.

Theorem 1 Let $X(t)$, $t = \ldots -1, 0, 1, \ldots$ be a P-dimensional stationary stochastic process whose covariance function and spectral density matrix satisfy conditions A and B of (1.4) and (1.5). Choose n, M and T integers so that $(2n+1)M = (T-1)/2$, and let $\{\hat{F}(\omega_\ell,X)\}_{\ell=1}^{M}$ be the sample spectral density matrices, defined by (1.10) based on X, the random matrix with i, s-th entry $X_i(s)$, $i = 1, 2, \ldots P$, $s = 1, 2, \ldots T$. Let $\{\hat{F}(\omega_\ell,\underset{\sim}{X})\}_{\ell=1}^{M}$ be the M independent complex Wishart $W_c(\bar{F}(\omega_\ell), P, 2n+1)$ matrices based on $\underset{\sim}{X}$, where $\underset{\sim}{X}$ is defined by (3.7).

Then

$$E\, \mathrm{Tr} \sum_{\ell=1}^{M} (\hat{F}(\omega_\ell,X) - \hat{F}(\omega_\ell,\underset{\sim}{X}))(\hat{F}(\omega_\ell,X) - \hat{F}(\omega_\ell,\underset{\sim}{X}))^{*'} \qquad (5.1)$$

$$\leq 8P^2(1+2\Lambda/\lambda)\theta^2 \log_2 M/(2\pi)^2(2n+1)^2$$

$$+ 12P(\Lambda/\lambda)\theta^2\, Mn/T^2$$

where θ, Λ and λ are given, respectively by (1.4), (1.5a) and (1.5b).

Proof: Let $\{\hat{F}(\omega_\ell,\tilde{X})\}_{\ell=1}^{M}$ be the M random matrices defined by (1.10) based on $\tilde{X}$, where $\tilde{X}$ is defined by (3.7). We prove

Lemma 5.1.

$$E\ \mathrm{Tr} \sum_{\ell=1}^{M} (\hat{F}(\omega_\ell,X) - \hat{F}(\omega_\ell,\tilde{X}))(\hat{F}(\omega_\ell,X) - \hat{F}(\omega_\ell,\tilde{X}))^{*'}$$
$$\leq P^2(1+2\Lambda/\lambda)4\theta^2 \log_2 M/(2\pi)^2(2n+1)^2 \tag{5.2}$$

and

Lemma 5.2

$$E\ \mathrm{Tr}(\hat{F}(\omega_\ell,\tilde{X}) - \hat{F}(\omega_\ell,\underset{\sim}{X}))(\hat{F}(\omega_\ell,\tilde{X}) - \hat{F}(\omega_\ell,\underset{\sim}{X}))^{*'} \leq 6P(\Lambda/\lambda)\theta n/T^2 \tag{5.3}$$

Applying Lemma A.1 of the Appendix to Lemmas 5.1 and 5.2 will then give the theorem.

In order to prove Lemma 5.1 we will need Lemma 5.3. Let $V \sim \eta(0,\Sigma)$, let A be any (real or complex) quadratic form with q^2 the largest eigenvalue of $AA^{*'}$. Let $\tilde{\Sigma}$ be a symmetric positive definite matrix of the same dimensions as Σ, let $\theta(\Sigma-\tilde{\Sigma}) \leq \theta$

and let Λ and λ be common upper and lower bounds for the eigen-values of $\sum$ and $\tilde{\sum}$, $0 < \lambda < \Lambda\ \infty$. Let $\tilde{V} = V\sum^{-1/2}\ \tilde{\sum}^{1/2}$. Then

$$E\ |V'AV - \tilde{V}'A\tilde{V}|^2 \leq (1+2\ \frac{\Lambda}{\lambda})\ q^2\theta^2 \tag{5.4}$$

The proof of this lemma is left to Appendix A.

Proof of Lemma 5.1.

Let $Q_\ell = Q_\ell(n,M,T)$ be the $T \times T$ matrix defined by

$$Q_\ell = \frac{1}{2\pi}\ \frac{1}{(2n+1)}\ WD_\ell W^*\ , \qquad \ell = 1,\ 2,\ \ldots\ M \tag{5.5}$$

where W is the unitary matrix given in Section 2, and $D_\ell = D_\ell(n,M,T)$ is the $T \times T$ diagonal matrix with ones in the rr-th position for $j_\ell - n \leq r \leq j_\ell + n$, $j_\ell = (\ell-1)(2n+1) + n+1$, and zeros elsewhere. Le $Q = Q(n,M,T,s_1,s_2,\ldots s_M)$ be defined by

$$Q = \sum_{\ell=1}^{M} s_\ell Q_\ell . \tag{5.6}$$

Then, from the definitions

$$\sum_{\ell=1}^{M} s_\ell(\hat{F}(\omega_\ell,X) - \hat{F}(\omega_\ell,\tilde{X})) = XQX' - \tilde{X}Q\tilde{X}' \tag{5.7}$$

Now let Q_{ij} be the PT × PT matrix with the T × T matrix Q in the ij-th block (of dimension T × T) and zeros elsewhere. We then have, from the definitions

$$\sum_{=1}^{M} \sum_{k=1}^{M} s_\ell s_k \; E \; \mathrm{Tr}(\hat{F}(\omega_\ell,X) - \hat{F}(\omega_\ell,\tilde{X}))(\hat{F}(\omega_\ell,X) - \hat{F}(\omega_\ell,\tilde{X}))'^{*}$$

$$= E \sum_{i,j=1}^{P} |(X_1:X_2:\ldots X_P)Q_{ij}(X_1:X_2:\ldots X_P)'$$

$$- (\tilde{X}_1:\tilde{X}_2:\ldots\tilde{X}_P)Q_{ij}(\tilde{X}_1:\tilde{X}_2:\ldots\tilde{X}_P)^{1}|^2 \qquad (5.8)$$

where X_i and $\tilde{X}_i$ are as before, the i-th rows of X and $\tilde{X}$ respectively.

Since, by construction, $\{2\pi(2n+1)Q_\ell\}_{\ell=1}^{M}$ are a family of orthogonal projections, the largest eigenvalue of QQ^* and hence $Q_{ij}Q_{ij}^*$ for every i, j, is bounded by $\max_\ell |s_\ell|^2/(2\pi)^2(2n+1)^2$. It may be shown, by an argument directly analogous to [5] p. 64, that the eigenvalues of $\sum$ are bounded above and below by Λ and λ, of (1.5), and this is clearly true for $\tilde{\sum}$. By Lemma 2.2 applied to each block of $(\sum-\tilde{\sum})$, we have $\phi(\sum-\tilde{\sum}) \leq 2\theta$ where θ is given by (1.4).We may therefore apply lemma 5.3 to each of the P^2 terms on the right of (5.6) to obtain

$$\sum_{=1}^{M} \sum_{k=1}^{M} s_\ell s_k \; E \; \mathrm{Tr}(\hat{F}(\omega_\ell,X) - \hat{F}(\omega_\ell,\tilde{X}))(\hat{F}(\omega_k,X) - \hat{F}(\omega_k,\tilde{X}))^{*'}$$

$$\leq P^2(1+2\Lambda/\lambda)4\theta^2 \max_\ell |s_\ell|^2/(2\pi)^2(2n+1)^2 \qquad (5.9)$$

Now let A be the (non-negative definite) $M \times M$ matrix with ℓ, k-th element $a_{\ell k}$ given by

$$a_{\ell k} = E\ \mathrm{Tr}(\hat{F}(\omega_\ell, X) - \hat{F}(\omega_\ell, \tilde{X}))(\hat{F}(\omega_k, X) - \hat{F}(\omega_k, \tilde{X}))^{*'} \qquad (5.10)$$

Then we may rewrite (5.9) as

$$\sum_{\ell=1}^{M} \sum_{k=1}^{M} s_\ell a_{\ell k} s_k \leq P^2(1+2\Lambda/\lambda)4\theta^2 \max_\ell |s_\ell|^2/(2\pi)^2(2n+1)^2 \qquad (5.11)$$

Using Lemma A.6 of the appendix, we see that (5.11) implies that

$$\mathrm{Tr}A = \sum_{=1}^{M} E\ \mathrm{Tr}(\hat{F}(\omega_\ell, X) - \hat{F}(\omega_\ell, \tilde{X}))(\hat{F}(\omega_\ell, X) - \hat{F}(\omega_\ell, \tilde{X}))^{*}$$

$$\leq P^2(1+2\Lambda/\lambda)4\theta^2 \log_2 M/(2\pi)^2(2n+1)^2. \qquad (5.12)$$

Proof of Lemma 5.2.

Let $\tilde{Z}(r)$ be the P-dimensional (column) vector given by the r-th column of $\tilde{X}W$, and, as in (4.8), let $\underset{\sim}{Z}(r)$ be the P-dimensional column vector given by the r-th column of $\underset{\sim}{X}W$. We have

$$\hat{F}(\omega_\ell, \tilde{X}) = \frac{1}{(2\pi)(2n+1)} \sum_{\nu=-n}^{n} \tilde{Z}(j_\ell+\nu)\tilde{Z}^{*'}(j_\ell+\nu) \qquad (5.13)$$

The density of (4.10) implies that $\tilde{Z}(r)$, $r = 1, 2, \ldots (T-1)/2$ are $(T-1)/2$ independent complex normal random vectors with

$E\ \tilde{Z}(r)\tilde{Z}^{*'}(r) = 2\pi F(\frac{2\pi r}{T})$. We have

$$\hat{F}(\omega_\ell, \underset{\sim}{X}) = \frac{1}{2\pi(2n+1)} \sum_{\nu=-n}^{n} \underset{\sim}{Z}(j_\ell+\nu)\ \underset{\sim}{Z}^{*'}(j_\ell+\nu) \tag{5.14}$$

where $\underset{\sim}{Z}(r)$ is as in Section 4 the r-th column of $\underset{\sim}{X}$ W. Using (3.6), it can be shown that

$$\underset{\sim}{Z}(r) = \bar{F}^{\frac{1}{2}}(\omega_\ell)\ F^{-\frac{1}{2}}(2\pi r/T)\ \tilde{Z}(r). \tag{5.15}$$

Letting G_r be the random matrix

$$G_r = \tilde{Z}(r)\tilde{Z}^{*'}(r) - \bar{F}^{\frac{1}{2}}(\omega_\ell)F^{-\frac{1}{2}}(2\pi r/T)\tilde{Z}(r)\tilde{Z}^{*'}(r)F^{-\frac{1}{2}}(2\pi r/T)\bar{F}^{\frac{1}{2}}(\omega_\ell) \tag{5.16}$$

we wish to show

$$E\ \mathrm{Tr}\ \frac{1}{(2\pi)^2(2n+1)^2} \left(\sum_{\nu=-n}^{n} G_{j_\ell+\nu}\right)\left(\sum_{\mu=-n}^{n} G^{*'}_{j_\ell+\mu}\right) \leq 6P(\Lambda/\lambda)\theta n/T^2. \tag{5.17}$$

Since $\tilde{Z}(r)$, $\tilde{Z}(s)$ are independent, $r \neq s \leq (T-1)/2$, we have

$$E\ \mathrm{Tr}G_r G_s^{*'} = (2\pi)^2 \mathrm{Tr}(F(\frac{2\pi r}{T}) - \bar{F}(\omega_\ell))(F(\frac{2\pi s}{T}) - \bar{F}(\omega_\ell))^{*'},\ r \neq s, \tag{5.18}$$

Furthermore, since, by the definition of $\bar{F}(\omega_\ell)$,

$$\frac{1}{(2n+1)^2} \mathrm{Tr} \sum_{\nu=j_\ell-n}^{j_\ell+n} \sum_{\mu=j_\ell-n}^{j_\ell+n} (F(\tfrac{2\pi\nu}{T}) - \bar{F}(\omega_\ell))(F(\tfrac{2\pi\mu}{T}) - \bar{F}(\omega_\ell))^{*'} = 0 \tag{5.19}$$

and, also,

$$\frac{1}{(2n+1)^2} \mathrm{Tr} \sum_{\nu=j_\ell-n}^{j_\ell+n} (F(\tfrac{2\pi\nu}{T}) - \bar{F}(\omega_\ell))(F\tfrac{2\pi\nu}{T}) - \bar{F}(\omega_\ell))^{*'} \geq 0 \tag{5.20}$$

we have

$$E\ \mathrm{Tr} \left(\frac{1}{(2\pi)(2n+1)} \sum_{\nu=-n}^{n} G_{j_\ell+\nu}\right) \left(\frac{1}{(2\pi)(2n+1)} \sum_{\mu=-n}^{n} G_{j_\ell+\mu}\right)^{*'}$$

$$\leq \frac{1}{(2\pi)^2(2n+1)^2} \sum_{\nu=-n}^{n} E\ \mathrm{Tr}\ G_{j_\ell+\nu} G_{j_\ell+\nu}^{*'} \tag{5.21}$$

Since $\tilde{Z}(r)$ is complex normal with $E\ \tilde{Z}(r)\ \tilde{Z}(r)^{*'} = 2\pi F(2\pi r/T)$ we may replace $\tilde{Z}(r)$ in (5.16) by

$$\tilde{Z}(r) = (2\pi)^{1/2} F^{1/2}(2\pi r/T)\xi(r), \quad r=1,2,\ldots(T-1)/2 \tag{5.22}$$

where $\xi(r)$, $r=1,2,\ldots(T-1)/2$ are independent 0 mean complex normal random vectors with $E\,\xi(r)\,\xi^{*'}(r) = I_{P\times P}$, where $I_{P\times P}$ is the $P \times P$ identity matrix. This substitution does not change the distribution of G_r.

We have

$$\text{Tr}\; G_r G_r^* = \tag{5.23}$$

$$(2\pi)^2 \text{Tr}\left(F^{1/2}(2\pi r/T)\xi(r)\xi(r)^{*'}F^{1/2}(2\pi r/T) - \bar{F}^{1/2}(\omega_\ell)\xi(r)\xi(r)^{*'}\bar{F}^{1/2}(\omega_\ell)\right) \times$$

$$\left(F^{1/2}(2\pi r/T)\xi(r)\xi(r)^{*'}F^{1/2}(2\pi r/T) - \bar{F}^{1/2}(\omega_\ell)\xi(r)\xi(r)^{*'}\bar{F}^{1/2}(\omega_\ell)\right)^{*'}$$

Now observe that, for $j_\ell - n \leq r \leq j_\ell + n$,

$$\phi(F(2\pi r/T) - \bar{F}(\omega_\ell)) = \sum_{i,j=1}^{P} |f_{ij}(2\pi r/T) - \bar{f}_{ij}(\omega_\ell)| \leq$$

$$\max_{|\omega-\omega_\ell| \leq (2\pi n/T)} \sum_{i,j=1}^{P} |f_{ij}(2\pi r/T) - f_{ij}(\omega)| \leq (2\theta n/T) \tag{5.24}$$

where θ is given by 1.4. This follows, since, for $|\omega - (2\pi r/T)| \leq (4\pi n/T)$,

$$|f_{ij}(2\pi r/T) - f_{ij}(\omega)| \leq \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} |R_{ij}(\tau)|\,|1 - e^{[(2\pi i r/T)-\omega]\tau}|$$

$$\leq (2n/T) \sum_{\tau=-\infty}^{\infty} |R_{ij}(\tau)|\,|\tau| \tag{5.25}$$

Using Lemma A.4 of appendix A yields the inequality

$$\text{Tr}\; G_r G_r^{*'} \leq (2\pi)^2 \;[\text{max eigenvalue } \xi(r)\xi(r)^{*'}]^2\; (\Lambda/\lambda)(2\theta n/T)^2 \tag{5.26}$$

Observing that $E||\xi(r)||^2 = 3P$, and putting together (5.21) and (5.26) gives the result.

6. Asymptotic behavior of $\lambda_i(X)$.

Let $\lambda_i(n,M,\underset{\sim}{X}) = \lambda_i(\underset{\sim}{X})$, $i = 1, 2, 3$ be defined by (4.12), (4.13) and (4.14), and let $\lambda_i(n,M,X) = \lambda_i(X)$ be the same statistics based on X, i.e. defined by (4.12), (4.13) and (4.14) with $\underset{\sim}{X}$ replaced by X. It will be shown in section 7, that, under the general alternative, $F(\omega)$ general, $F(\omega)$ and $R(\tau)$ satisfy conditions A and B, that $\text{var}(\log \lambda_i(\underset{\sim}{X})) = 0(M/n)$, $i = 1, 2, 3$. It will be shown that, under H_i, $\text{var}(\log \lambda_i(\underset{\sim}{X})) = 0(M/n^2)$ $i = 1, 2$. Under H_3, we may, without loss of generality, take $P = 1$ and $\Sigma = I$. In this case $\Sigma = \underset{\sim}{\Sigma}$ and $\lambda_3(X) = \lambda_3(\underset{\sim}{X})$. We have

Theorem 2. As $n,M \to \infty$ in such a way that $(\log_2 M)/n \to 0$,

$$E(n/M)^{1/2} \; |\log \lambda_i(\underset{\sim}{X}) - \log \lambda_i(X)| \to 0, \quad i = 1, 2, 3. \tag{6.1}$$

If H_i, $i = 1, 2$ are true, then

$$E(n/M^{1/2}) \; |\log \lambda_i(\underset{\sim}{X}) - \log \lambda_i(X)| \to 0, \quad i = 1, 2. \tag{6.2}$$

The proof will be carried out for $i = 2$. The proof in the other cases is carried out in the same way. Define $\hat{W}_{1.2,3...P}(\omega_\ell,X)$, the sample multiple coherence between the first and the other P-1 components of X by

$$\hat{W}_{1.2,3,\ldots P}(\omega_\ell,Y) = \hat{W}(\omega_\ell,y) = \hat{F}_{12}(\omega_\ell,y)\hat{F}_{22}^{-1}(\omega_\ell,Y)\hat{F}_{12}^{*'}(\omega_\ell,y)/\hat{f}_{11}(\omega_\ell,Y) \tag{6.3}$$

where

$$\hat{F}(\omega_\ell,y) = \left(\begin{array}{c|c} \hat{f}_{11}(\omega_\ell,y) & \hat{F}_{12}(\omega_\ell,Y) \\ \hline \hat{F}_{12}(\omega_\ell,Y)^{*'} & F_{22}(\omega_\ell,y) \end{array}\right)$$

$\hat{W}(\omega_\ell,y)$ satisfies [see 4]

$$\frac{|F(\omega_\ell,y)|}{\hat{f}_{11}(\omega_\ell,y)\,|\hat{F}_{22}(\omega_\ell,y)|} = 1 - \hat{W}(\omega_\ell,y) \geq 0 \tag{6.4}$$

Using the fact that, for $0 \leq \mu,\ \nu < 1$, $|\log(1-u) - \log(1-\nu)| \leq |u-\nu|\ \left|\frac{1}{1-u} + \frac{1}{1-\nu}\right|$ gives

$$E(n/M)^{1/2} \; |\log \lambda_2(\underset{\sim}{X}) - \log \lambda_2(X)|$$

$$\leq (n/M)^{1/2} \sum_{\ell=1}^{M} E|\log(1-\hat{W}(\omega_\ell,\underset{\sim}{X}) - \log(1-\hat{W}(\omega_\ell,X))|$$

$$\leq (n/M)^{1/2} \sum_{\ell=1}^{M} E\; g(\omega_\ell) \sum_{i,j=1}^{P} |\hat{f}_{ij}(\omega_\ell,\underset{\sim}{X}) - \hat{f}_{ij}(\omega_\ell,X)|$$

$$\leq (n/M)^{1/2} \sum_{\ell=1}^{M} [E\; g^2(\omega_\ell)]^{1/2}$$

$$[E\; Tr(\hat{F}(\omega_\ell,\underset{\sim}{X}) - \hat{F}(\omega_\ell,X))(\hat{F}(\omega_\ell,\underset{\sim}{X}) - \hat{F}(\omega_\ell,X))^{*'}]^{1/2} \tag{6.5}$$

where $E\; g^2(\omega_\ell)$ is bounded by a constant depending only on $F(\omega)$. We can thus conclude that

$$E(n/M)^{1/2} \; |\log \lambda_2(\underset{\sim}{X}) - \log \lambda_2(X)|$$

$$\leq \text{const. } n^{1/2} \; [\sum_{\ell=1}^{M} E(Tr(\hat{F}(\omega_\ell,\underset{\sim}{X}) - \hat{F}(\omega_\ell,X))(\hat{F}(\omega_\ell,\underset{\sim}{X}) - \hat{F}(\omega_\ell,X))]^{1/2}$$

$$\leq \text{const } n^{1/2} \; [\log M/(2n+1)^2 + Mn/T^2]^{1/2} \to 0 \tag{6.6}$$

When H_2 is true $\sum_{j=2}^{P} |f_{1j}(\omega_\ell,\underset{\sim}{X})| + |\hat{f}_{1j}(\omega_\ell,X)|$ is factored out of $g(\omega_\ell)$ to get an expression of the form

$$E(n/M^{1/2})\ |\log \lambda_2(\underset{\sim}{X}) - \log \lambda_2(X)|$$

$$\leq \text{const}\ (n/M^{1/2} \sum_{\ell=1} [E\ h^4(\omega_\ell)]^{1/4}\ E(\sum_{j=2}^{p} |\hat{f}_{ij}(\omega_\ell,\underset{\sim}{X})|^4 + |\hat{f}_{ij}(\omega_\ell,X)|^4)^{1/4}.$$

$$\cdot\ E\ \mathrm{Tr}(\hat{F}(\omega_\ell,\underset{\sim}{X}) - \hat{F}(\omega_\ell,X))(\hat{F}(\omega_\ell,\underset{\sim}{X}) - \hat{F}(\omega_\ell,X))^{1/2} \tag{6.7}$$

where $E\ h^4(\omega_\ell)$ is bounded by a constant depending only on $F(\omega)$. Using the facts that $\hat{f}_{1j}(\omega_\ell,\underset{\sim}{X})$ and $\hat{f}_{1j}(\omega_\ell,X)$ are quadratic forms in normal random variables, and for such quadratic forms t, $E\ t^4 \leq c\ (E\ t^2)^2$ where c is a universal constant, and, under the null hypothesis $E\ |\hat{f}_{1j}(\omega_\ell,\underset{\sim}{X})|^2$ and $E\ |\hat{f}_{1j}(\omega_\ell,X)|^2$ are bounded by a constant $\times$ $1/(2n+1)$, the result is

$$E\ (n/M^{1/2})\ |\log \lambda_2(\underset{\sim}{X}) - \log \lambda_2(X)|$$

$$\leq \text{const}\ (n/M^{1/2})\ n^{-1/2}\ [\log M/(2n+1)^2 + Mn/T^2]^{1/2} \to 0. \tag{6.8}$$

7. Distribution Theory

7.1 Distribution theory for $\lambda_1(\underset{\sim}{X})$ and $\lambda_2(\underset{\sim}{X})$.

Let

$$\underset{\sim}{\lambda}_{1\ell} = |\hat{F}(\omega_\ell,\underset{\sim}{X})|\ /\ \prod_{j=1}^{P} \hat{f}_{jj}(\omega_\ell,\underset{\sim}{X}) \tag{7.1}$$

$$\underset{\sim}{\lambda}_{2\ell} = |\hat{F}(\omega_\ell,\underset{\sim}{X})| \,/\, \hat{f}_{11}(\omega_\ell,\underset{\sim}{X}) \; |\hat{F}_{22}(\omega_\ell,\underset{\sim}{X})| \qquad (7.2)$$

$$\ell=1,2,\ldots M$$

and let $\lambda_i(\underset{\sim}{X}) = \underset{\sim}{\lambda}_i = \prod_{\ell=1}^{M} \underset{\sim}{\lambda}_{i\ell}$, $i=1,2$.

Suppose $\hat{S} = \{\hat{S}_{ij}\}$ is a $P \times P$ real Wishart matrix estimated on n'degrees of freedom, say a sample covariance matrix,

$$\hat{S} \sim W(\Sigma,P,n')$$

Let

$$\hat{S} = \begin{pmatrix} \hat{s}_{11} & \hat{s}_{12} \\ \hat{s}_{12} & \hat{S}_{22} \end{pmatrix}$$

As is well known, [See 1], the likelihood ratio statistics η_1 and η_2 for the hypotheses

A_1 : S diagonal

A_2 : $s_{ij} = 0 \quad j = 2, 3, \ldots P, \quad \Sigma = \{\sigma_{ij}\}$

are, respectively

$$\eta_1 = |\hat{S}| \,/\, \prod_{i=1}^{P} \hat{s}_{ii}$$

$$\eta_2 = |\hat{S}| \,/\, \hat{s}_{11} \, |\hat{S}_{22}|$$

And, when A_1 and A_2, respectively are true, it is well known that λ_1 and λ_2 are distributed as

$$\lambda_1 \sim \prod_{j=1}^{P-1} \beta_{\frac{1}{2}(n'-j),\frac{1}{2}j}$$

$$\lambda_2 \sim \beta_{\frac{1}{2}(n'-(P-1)),\ \frac{1}{2}(P-1)}$$

where $\beta_{\mu,\nu}$ are independent Beta random variables with μ and ν degrees of freedom. By the same techniques used in [1] it is straight forward to show, that, when H_1, H_2, respectively, are true,

$$\underset{\sim}{\lambda}_{1\ell} \sim \prod_{j=1}^{P-1} \beta_{n'-j,j} \qquad \ell = 1, 2, \ldots M \tag{7.3}$$

$$\underset{\sim}{\lambda}_{2\ell} \sim \beta_{n'-(P-1),(P-1)} \qquad \ell = 1, 2, \ldots M \tag{7.4}$$

where we have written $n' = 2n+1$. (The distribution of $\underset{\sim}{\lambda}_{2\ell}$ under the null and alternative is given in [4] by Goodman.) Since the characteristic function $\phi_{\mu,\nu}(s)$ of $-\log \beta_{\mu,\nu}$ is

$$\Phi_{\mu,\nu}(s) = \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)} \frac{\Gamma(\mu-is)}{\Gamma(\mu+\nu-is)}, \tag{7.5}$$

and $\{\underset{\sim}{\lambda}_{1\ell}\}_{\ell=1}^{M}$ are independent, we have that the characteristic function $\Phi_1(s)$ of $t_1 = -\frac{1}{M}\log \underset{\sim}{\lambda}_1$ is given by

$$\Phi_1(s) = C_{M,n',P} \prod_{j=1}^{P-1} \frac{\Gamma(n'-j-\frac{is}{M})^M}{\Gamma(n'-\frac{is}{M})^M}$$

$$= C_{M,n',P} \left[\prod_{j=1}^{P-1} \Gamma(n'-j-\frac{is}{M})^{(P-j)M} \right]^{-1} \qquad (7.6)$$

$$C_{M,n',P} = \frac{\Gamma(n')^{(P-1)M}}{\prod_{j=1}^{P-1} \Gamma(n'-j)^{M}} .$$

$\Phi_1(s)$ may be inverted by standard formulae [2]. For example, for $P = 3$ the density $f_1(x)$ for $t_1 = -\frac{1}{M}\log \underset{\sim}{\lambda}_1$ is

$$f_1(x) = (n'-1)^{2M}(n'-2)^M \left[\sum_{\ell=1}^{2M} \frac{(-1)^M}{(2M-\ell)!} \binom{M+\ell-2}{\ell-1} \frac{1}{M}\left(\frac{x}{M}\right)^{2M-\ell} e^{-(n'-1)\frac{x}{M}} \right.$$

$$\left. + \sum_{\ell=1}^{M} \frac{(-1)^{M-1}}{(M-\ell)!} \binom{2M+\ell-2}{\ell-1} \frac{1}{M}\left(\frac{x}{M}\right)^{M-\ell} e^{-(n'-2)\frac{x}{M}} \right] , \quad x \geq 0$$

$$= 0, \; x < 0 . \qquad (7.7)$$

Similarly, the characteristic function $\phi_2(s)$ of $t_2 = -\log \underset{\sim}{\lambda}_2$ is given by

$$\Phi_2(s) = D_{M,n',P} \left[\frac{\Gamma(n'-(P-1)-\frac{is}{M})}{\Gamma(n'-\frac{is}{M})} \right]^M$$

$$= D_{M,n',P} \prod_{j=1}^{P-1} \Gamma(n'-j-\frac{is}{M})^{-M} \quad . \qquad (7.8)$$

Where

$$D_{M,n',P} = \frac{\Gamma(n')^M}{\Gamma(n'-(P-1)^M}$$

The density $f_2(x)$ for $t_2 = -\frac{1}{M} \log \lambda_2$ may be formed by the same formulae, the result for $P = 3$ is

$$f_2(x) = (n'-1)^M (n'-2)^M \sum_{\ell=1}^{M} \frac{1}{(M-\ell)!} \binom{M+\ell-2}{\ell-1} \frac{1}{M} (\frac{x}{M})^{M-\ell}$$

$$\times \left[(-1)^M e^{-(n'-1)\frac{x}{M}} + (-1)^{\ell-1} e^{-(n'-2)\frac{x}{M}} \right] . \qquad (7.9)$$

Using (6.6) and (6.8) the null means and variances of t_1 and t_2 are given by

$$E\ t_1 = \sum_{j=1}^{P-1} \frac{(P-j)}{(n'-j)}$$

$$\mathrm{Var}\ t_1 = \frac{1}{M} \sum_{j=1}^{P-1} \frac{(P-j)}{(n'-j)^2} = 0\left(\frac{1}{Mn'^2}\right) \qquad (6.10)$$

$$E\ t_2 = \sum_{j=1}^{P-1} \frac{1}{(n'-j)}$$

$$\mathrm{Var}\ t_2 = \frac{1}{M} \sum_{j=1}^{P-1} \frac{1}{(n'-j)^2} = 0\left(\frac{1}{Mn'^2}\right)$$

Choose n large fixed, then let $M \to \infty$, t_1 and t_2 are both then asymptotically normal by the central limit theorem. Expressions for the mean and variance of t_2 under the alternative may be obtained using the density of $\underset{\sim}{\lambda}_{2\ell}$ given by Goodman. The expressions are unwieldy and will not be reproduced here.

A simpler procedure seems to be to consider M large fixed, and let $n' \to \infty$. Then, under the general alternative to H_1, we have, by Theorem 4.2.5 [1], that

$$\sqrt{n'} \left[-\frac{1}{M} \log \underset{\sim}{\lambda}_1 + \frac{1}{M} \sum_{k=1}^{M} \log \frac{|\bar{F}(\omega_k)|}{\prod_{i=1}^{P} \bar{f}_{11}(\omega_k)} \right] \qquad (7.11)$$

is asymptotically normal with zero mean. Carrying out the differentiation required by that theorem, it is possible to show that the asymptotic variance is

$$\frac{1}{M^2} \sum_{\ell=1}^{M} \sum_{i \neq j} |\bar{W}_{ij}(\omega_\ell)| \approx \frac{1}{2\pi M} \int_0^{2\pi} \sum_{i<j} |W_{ij}(\omega)| d\omega \tag{6.12}$$

where $\bar{W}_{ij}(\omega_\ell) = |\bar{f}_{ij}(\omega_\ell)|^2 / \bar{f}_{ii}(\omega_\ell)\ \bar{f}_{ij}(\omega_\ell)$ and $W_{ij}(\omega)$, the coherence between the two time series $X_1(t)$ and $X_2(t)$ is given by $W_{ij}(\omega) = |f_{ij}(\omega)|^2 / f_{ii}(\omega)\ f_{jj}(\omega)$.

Under the general alternative to H_2, for M(large) fixed, we have, again by Theorem 4.2.5, [1], as $n' \to \infty$, that

$$\sqrt{n'} \left[-\frac{1}{M} \log \lambda_2 + \frac{1}{M} \sum_{k=1}^{M} \log |\bar{W}_{1.2,3,\ldots,p}(\omega_k)| \right] \tag{7.13}$$

is asymptotically normal for large n', with 0 mean, where $\bar{W}_{1.2,3,\ldots p}(\omega)$ is given by

$$\bar{W}_{1.2,3,\ldots p}(\omega) = |\bar{F}(\omega)| \;/\; \bar{f}_{11}(\omega)\ |\bar{F}_2(\omega)| \;. \tag{7.14}$$

$$\bar{F}(\omega) = \begin{pmatrix} \bar{f}_{11}(\omega) & \bar{F}_{12}(\omega) \\ \bar{F}^{*\prime}_{12}(\omega) & \bar{F}_2(\omega) \end{pmatrix}$$

The asymptotic variance can be shown to be

$$\frac{2}{M^2} \sum_{\ell=1}^{M} \bar{W}_{1.2,3,\ldots p}(\omega_\ell) \approx \frac{1}{2\pi M} \int_0^{2\pi} W_{1.2,3,\ldots p}(\omega)\, d\omega, \tag{7.15}$$

where $W_{1.2,3,...P}(\omega_\ell)$, the partial coherence between $X_1(t)$ and $\{X_1(t),\ j = 2,\ 3,\ \dots\ P\}$ is given by $W_{1.2,3,...P}(\omega_\ell) = |F(\omega_\ell)| \ / \ f_{11}(\omega_\ell) \ \ F_2(\omega_\ell)|$,

$$F(\omega_\ell) = \left(\begin{array}{c|c} f_{11}(\omega_\ell) & F_{12}(\omega_\ell) \\ \hline F_{12}^{*'}(\omega_\ell) & F_{22}(\omega_\ell) \end{array}\right)$$

7.2 Distribution Theory for $\lambda_3(\underset{\sim}{X})$

Let

$$\underset{\sim}{\lambda}_3 = \lambda_3(\underset{\sim}{X}) = \frac{\prod_{\ell=1}^{M} \hat{f}_{11}(\omega_\ell,\underset{\sim}{X})}{(\frac{1}{M}\sum_{\ell=1}^{M} \hat{f}_{11}(\underset{\sim}{X}))^M}$$

where, according to (4.5) and the properties of complex Wishart matrices, we have $\{2(2n+1)\ \hat{f}_{11}(\omega_\ell,\underset{\sim}{X}) \ / \ \bar{f}_{11}(\omega_\ell),\ \ell = 1,2,\dots M\}$ are distributed independently as $\chi^2_{2(2n+1)}$. Consider M in dependent populations, the ℓ-th population being $\mathcal{N}(\mu_\ell, s_\ell^2)$, $\ell = 1,2,\dots M$. Estimate s_ℓ^2 by $\hat{s}_\ell^2$, the sample variance, estimated with $2(2n+1)$ degrees of freedom. Then V, where

$$V = \left[\frac{\prod_{\ell=1}^{M} \hat{s}_\ell^2}{(\sum_{\ell=1}^{M} \hat{s}_\ell^2)^M}\right]^{2n+1} \tag{7.16}$$

is, except for a constant, the well known is Bartlett's statistic for homogeneity of variances. Hence $\underset{\sim}{\lambda}_3$ is distributed as $M^M V^{\frac{1}{2n+1}}$. The joint density of $\hat{f}_{11}(\omega_\ell, \underset{\sim}{X}) = \underset{\sim}{f}_\ell$, $\ell = 1,2,\ldots M$ is

$$P(\underset{\sim}{f}_1, \underset{\sim}{f}_2, \ldots \underset{\sim}{f}_M) = \prod_{\ell=1}^{M} \frac{(\underset{\sim}{f}_\ell)^{n'-1}}{\Gamma(n')\ \bar{f}_\ell^{\,n'}} e^{-\underset{\sim}{f}_\ell/\bar{f}_\ell} \tag{7.17}$$

where $n' = 2n+1$ and $\bar{f}_\ell \equiv \bar{f}_{11}(\omega_\ell)$.

Now let $c_\ell = 1/\bar{f}$, $\bar{c} = \frac{1}{2}[\max_\ell c_\ell + \min_\ell c_\ell]$ and let $\xi_\ell = \bar{c}\ \underset{\sim}{f}_\ell$, $b_\ell = (\bar{c}-c_\ell)/\bar{c}$, where $|b_\ell|<1$, $\ell = 1,2,\ldots M$. Then, since the distribution of $\underset{\sim}{\lambda}_3$ is invariant under a constant multiplicative factor applied to $\underset{\sim}{f}_\ell$, $\log \underset{\sim}{\lambda}_3$ is distributed as

$$\log \left(\prod_{\ell=1}^{M} \xi_\ell\right) / \left(\frac{1}{M} \sum \xi_\ell\right)^M, \tag{7.18}$$

where the joint density of $\{\xi_\ell\}_{\ell=1}^{M}$ is given by

$$P(\xi_1, \xi_2, \ldots \xi_M) = \frac{1}{[\Gamma(n')]^M} \prod_{k=1}^{M} (1-b_k)^{n'} \prod_{\ell=1}^{M} \xi_\ell^{n'-1}\ e^{-\sum_{\nu=1}^{M} \xi_\nu + \sum_{\nu=1}^{M} b_\nu \xi_\nu} \tag{7.19}$$

In appendix B it is shown that the characteristic function $\Phi_3(s)$ of $\log \underset{\sim}{\lambda}_3$ is given by

$$\Phi_3(s)\ E\ e^{is \log \underset{\sim}{\lambda}_3} = \Phi_o(s)\ g(s)$$

where

$$\Phi_o(s) = M^{isM} \frac{\Gamma(Mn')}{\Gamma(n')M} \frac{\Gamma(is+n')^M}{\Gamma((is+n')M)} \tag{7.20a}$$

is the characteristic function of log $\underset{\sim}{\lambda}_3$ under the null hypothesis and

$$g(s) = \prod_{\nu=1}^{M} (1-b_\nu)^{n'} \sum_{k=0}^{\infty} \psi_k(s)\theta_k(s) \tag{7.20b}$$

where

$$\psi_k(s) = \frac{\Gamma((is+n')M)}{\Gamma((is+n')M+k)} \frac{\Gamma(Mn'+k)}{\Gamma(Mn)} \tag{7.20c}$$

and $\theta_k(s)$ is defined by

$$\sum_{k=0}^{\infty} \theta_k(s)t^k = \prod_{\nu=1}^{M} (1-b_\nu t)^{-(is+n')} \qquad |t| < 1 \tag{7.20d}$$

$\Phi_o(s)$ for $n = 1$ was given by Whittle in 1950, [10] where he suggests the use of $\underset{\sim}{\lambda}_3$ (with $n = 1$) as a statistic for testing the independence of residuals after model fitting. Using the duplication formula for the gamma function [9]

$$\Gamma(Mz) = [M^{\frac{1}{2}-Mz} (2\pi)^{\frac{1}{2}(M-1)}]^{-1} \Gamma(z)\,\Gamma(z+\tfrac{1}{M}) \ldots \Gamma(z+\tfrac{M-1}{M})$$

it follows from (7.5) and (7.20a) that $\underset{\sim}{\lambda}_3$ under the null hypothesis, is distributed as

$$\underset{\sim}{\lambda}_3 \sim \prod_{j=1}^{M} \beta_{n', \frac{j}{M}} \qquad (7.21)$$

where $\{\beta_{n',j/M}\}_{i=0}^{M}$ are M independent $\beta_{n',j/M}$ random variables.

The moments of log $\underset{\sim}{\lambda}_3$, under the null hypothesis are readily obtainable by using the formula for the logarithmic derivative of the gamma function [9], which gives

$$\frac{\partial^\ell}{\partial s^\ell}\bigg|_{s=0} \log \frac{\Gamma(is+n)^M}{\Gamma((is+n)M)} = (-iM)^\ell (\ell-1)!\,[\frac{1}{M^{\ell-1}} \sum_{j=0}^{\infty} \frac{1}{(n+j)^\ell} - \sum_{j=0}^{\infty} \frac{1}{(Mn+j)^\ell}] \qquad (7.22)$$

Although Bartletts statistic has been with us since 1937, its distribution theory does not seem to be in a completely satisfactory state. Let $W_1 = \underset{\sim}{\lambda}_3^{n'}$. An asymptotic expansion, for the c.d.f. of log W_1, for large n', in terms of χ^2 c.d.f,'s is given in Anderson [1] (p-225, Eq. (7).), which should suffice for practical purposes.

For sufficiently small b_ν, log g(s) can be expanded up to the second order in $\{b_\nu\}$ as

$$\log g(s) = \log \prod_{\nu} (1-b_\nu)^{n'} + \log(1 + \sum_{1}^{\infty} \psi_k(s)\theta_k(s))$$

$$\simeq \text{constant} + \psi_1(s)\theta_1(s)+\psi_2(s)\theta_2(s)+\ldots-\tfrac{1}{2}[\psi_1(s)\theta_1(s)]^2+\ldots$$

$$= \text{constant} + \frac{(Mn+1)Mn}{2M((is+n')M+1)} \sum_{\nu=1}^{M} (b_\nu-\bar{b})^2 \tag{7.23}$$

where $\bar{b} = \frac{1}{M} \sum b_\nu$. Hence, for $|b_\nu|$ sufficiently small,

$$E \log \lambda_3 \simeq \mu_o - \frac{M^2 n}{2(Mn+1)} \left(\frac{1}{M} \sum_{\nu=1}^{M} (b_\nu-\bar{b})^2\right)$$

$$\text{Var} \log \lambda_3 \simeq \sigma_o^2 + \frac{M^3 n}{(Mn+1)^2} \left(\frac{1}{M} \sum_{\nu=1}^{M} (b_\nu-\bar{b})^2\right) = 0(M/n) \tag{7.24}$$

where μ_o and σ_o^2, the null mean and variance are, from (7.22),

$$\mu_o = M(\log M - \sum_{j=n}^{Mn-1} \frac{1}{j})$$

$$\sigma_o^2 = M \sum_{j=1}^{\infty} \frac{1}{(n+j)^2} - M^2 \sum_{j=0}^{\infty} \frac{1}{(Mn+j)^2}$$

It is possible to show that the neglected terms in (7.24) are $0(M/n)$ times higher order terms in $\{b_\nu\}$, the details are omitted.

For sufficiently small $\{b_\nu\}$,

$$\frac{1}{M} \sum_{\nu=1}^{M} (b_\nu - \bar{b})^2 \simeq \frac{1}{2\pi} \int_0^{2\pi} ([f_{11}(\omega) - \bar{f}]^2 \; / \; (\bar{f})^2) \, d\omega \tag{7.25}$$

where

$$\bar{f} = \frac{1}{2\pi} \int_0^{2\pi} f(\omega) \, d\omega. \tag{7.26}$$

A detailed examination of the power of this test against near by alternatives might begin with equation (B.6), where $\Phi_3(s)$ is expressed as a weighted sum of characteristic functions, the weights being a constant multiple of terms in the power series expansion of $\prod_{\nu=1}^{M} (1-b_\nu)^{n'}$. This investigation is omitted.

Appendix A

Lemmas A.5 and A.6 are used in the proof of Theorem 1. Lemmas A.1-1.3 are used in the proof of Lemma A.4, which is needed for Lemma A.5.

In Lemmas A.1-A.3, A and B are strictly positive definite matrices, U, U_i and V are any square matrices, and λ_Z, Λ_Z are the smallest and largest eigenvalues of the matrix Z. The fact that $\lambda_A \mathrm{Tr}\ B \leq \mathrm{Tr}\ AB \leq \Lambda_A \mathrm{TR}\ B$ is repeatedly used.

Lemma A.1

$$\mathrm{Tr}\left(\sum_{j=1}^{N} U_j\right)\left(\sum_{j=1}^{N} U_j\right)^{*'} \leq N\ \mathrm{Tr}\left(\sum_{j=1}^{N} U_j U^{*'}\right).$$

Proof.

$$\begin{aligned}\mathrm{Tr}\left(\sum_{j=1}^{N} U_j\right)\left(\sum_{j=1}^{N} U_j\right)^{*'} &= \mathrm{Tr} \sum_{j=1}^{N}\sum_{k=1}^{N} U_j U_k^{*'} \\ &\leq \mathrm{Tr} \sum_{j=1}^{N}\sum_{k=1}^{N} \frac{1}{2}(U_j U_j^{*'} + U_k U_k^{*'}) \\ &= N\ \mathrm{Tr}\left(\sum_{j=1}^{N} U_j U_j^{*'}\right) .\end{aligned}$$

Lemma A.2

$$\mathrm{Tr}(A-B)^2 \le \frac{1}{(\lambda_A+\lambda_B)^2}\,\mathrm{Tr}(A^2-B^2)^2 \quad .$$

Proof.

$$(\lambda_A+\lambda_B)\ \mathrm{Tr}(A-B)^2 \le \mathrm{Tr}(A-B)^2(A+B) = \mathrm{Tr}(A-B)(A+B)(A-B)$$

$$(\lambda_A+\lambda_B)\ \mathrm{Tr}(A-B)(A+B)(A-B) \le \mathrm{Tr}(A-B)(A+B)(A-B)(A+B)$$

giving

$$(\lambda_A+\lambda_B)^2\ \mathrm{Tr}(A-B)^2 \le \mathrm{Tr}(A-B)(A+B)(A-B)(A+B).$$

$$\mathrm{Tr}(A-B)(A+B)(A-B)(A+B) = \mathrm{Tr}(A^2-BA+AB-B^2)(A^2-BA+AB-B^2)$$

$$= \mathrm{Tr}[A^4 - A^2BA + A^3B - A^2B^2 - BA^3 + BABA - BAAB + BAB^2 + ABA^2$$

$$- ABBA + ABAB - AB^3 - B^2A^2 + B^3A - B^2AB + B^4]$$

$$= \mathrm{Tr}[A^4 - 4B^2A^2 + 2BABA + B^4].$$

Now,

$$\mathrm{Tr}\ BABA \le \mathrm{Tr}\ A^2B^2 \ ,$$

hence

$$\mathrm{Tr}[A^4-4B^2A^2+2BABA+B^4] \le \mathrm{Tr}[A^4-2\ \mathrm{Tr}\ A^2B^2+B^4] = \mathrm{Tr}[A^2-B^2]^2 \ .$$

Combining these last inequalities gives the lemma. It is easy to see that equality is obtained for $A = aI$, $B = bI$.

Lemma A.3

$$\mathrm{Tr}(U-V)(U-V)^{*'} \leq [\phi(U-V)]^2 .$$

Proof.

$$\mathrm{Tr}(U-V)(U-V)^{*'} = \sum_{\mu,\nu} (u_{\mu\nu}-v_{\mu\nu})^2 - [\sum_{\mu,\nu} |u_{\mu\nu}-v_{\mu\nu}|]^2 = [\phi(U-V)]^2.$$

Lemma A.4 Let A be any (real or complex) quadratic form with q^2 the largest eigenvalue of $AA^{*'}$. Let S, $\tilde{S}$ be strictly positive definite (real or complex) matrices of the same dimension as A and let $0 < \lambda \leq \Lambda < \infty$ be common lower and upper bounds for the eigenvalues of S and $\tilde{S}$, and suppose $\phi(S-\tilde{S})$, defined in Lemma 2 satisfies $\phi(S-\tilde{S}) \leq \theta$. Then

$$\mathrm{Tr}(S^{1/2}AS^{1/2}-\tilde{S}^{1/2}A\tilde{S}^{1/2})\ (S^{1/2}AS^{1/2}-\tilde{S}^{1/2}A\tilde{S}^{1/2})^{*'} \leq \frac{\Lambda}{\lambda} q^2\theta^2 .$$

Proof.

$$\mathrm{Tr}(S^{1/2}AS^{1/2}-\tilde{S}^{1/2}A\tilde{S}^{1/2})(S^{1/2}AS^{1/2}-\tilde{S}^{1/2}A\tilde{S}^{1/2})^{*'}$$

$$= \mathrm{Tr}[(S^{1/2}-\tilde{S}^{1/2})AS^{1/2}+\tilde{S}^{1/2}A(S^{1/2}-\tilde{S}^{1/2})][(S^{1/2}-\tilde{S}^{1/2})AS^{1/2}+\tilde{S}^{1/2}A(S^{1/2}-\tilde{S}^{1/2})]^{*'},$$

which by Lemma A.1 is less than

$$2 \operatorname{Tr}(S^{1/2}-\tilde{S}^{1/2})(S^{1/2}-\tilde{S}^{1/2})^{*'} (ASA^{*'}+A\tilde{S}A^{*'})$$

$$\leq 4q^2 \operatorname{Tr}(S^{1/2}-\tilde{S}^{1/2})(S^{1/2}-\tilde{S}^{1/2})^{*'}.$$

Lemmas A.2 and A.3 give

$$\operatorname{Tr}(S^{1/2}-\tilde{S}^{1/2})(S^{1/2}-\tilde{S}^{1/2})^{*'}$$

$$\leq \frac{1}{4\lambda} \operatorname{Tr}(S-\tilde{S})(S-\tilde{S})^{*'} \leq \frac{1}{4\lambda} [\phi(S-\tilde{S})]^2 = \frac{1}{4\lambda} \theta^2,$$

which gives the result.

<u>Lemma A.5.</u> Let $Y \sim \mathcal{N}(0,S)$, let A be any (real or complex) quadratic form with q^2 the largest eigenvalue of $AA^{*'}$. Let $\tilde{S}$ be a symmetric positive definite matrix of the same dimensions as S, let $\phi(S-\tilde{S}) \leq \theta$, and let Λ and λ be common upper and lower bounds for the eigenvalues of S and $\tilde{S}$, $0 < \lambda < \Lambda < \infty$. Let $\tilde{Y} = YS^{-1/2}\tilde{S}^{1/2}$. Then

$$E|YAY'-\tilde{Y}A\tilde{Y}'|^2 \leq (1 + 2 \tfrac{\Lambda}{\lambda})q^2\theta^2 .$$

<u>Proof.</u>

$$|YAY'-\tilde{Y}A\tilde{Y}'|^2 = |Y(A-S^{-1/2}\tilde{S}^{1/2}A\tilde{S}^{1/2}S^{-1/2})Y'|^2 .$$

Letting $H = A - S^{-1/2}\tilde{S}^{1/2}A\tilde{S}^{1/2}S^{-1/2}$, we have, using the relations between 4th order mixed moments of Gaussian random variables

$$E|YHY'|^2 = (\text{Tr } HS)^2 + 2 \text{ Tr } HSH'S, \tag{A.5.1}$$

$$(\text{Tr } HS)^2 = (\text{Tr } A(S-\tilde{S}))^2 \le [\text{largest absolute entry of } A \times \phi(S-\tilde{S})]^2 \le q^2\theta^2 , \tag{A.5.2}$$

$$\begin{aligned} \text{Tr } HSH'S &= \text{Tr}(A-S^{-1/2}\tilde{S}^{1/2}A\tilde{S}^{1/2}S^{-1/2})S(A-S^{-1/2}\tilde{S}^{1/2}A\tilde{S}^{1/2}S^{-1/2})'S \\ &= \text{Tr}(S^{1/2}AS^{1/2}-S^{1/2}AS^{1/2})(S^{1/2}AS^{1/2}-S^{1/2}AS^{1/2})'. \end{aligned} \tag{A.5.3}$$

Using Lemma A.4 on the right-hand side of (A.5.3) yields the inquality

$$\text{Tr } HSH'S \le \frac{\Lambda}{\lambda} q^2\theta^2 .$$

Combining (A.5.1), (A.5.2) and (A.5.4) gives the result.

Lemma A.6 Let A be a non-negative definite $M \times M$ matrix, and suppose it is known that, for any $s = (s_1, s_2, \ldots s_M)$, $s A s' \le \max_{\ell} |s_\ell|^2 c^2$. Then, if $M = 2^k$ for some k, $\text{Tr } A \le c^2$. In general, $\text{Tr } A \le (\log_2 M)\, c^2$.

Proof. We use the fact, that if $y^1, y^2, \ldots y^M$ are any orthonormal set of M dimensional (row) vectors, then $\text{Tr } C = \sum_{k=1}^{M} y^k C y^{k'}$. If

$M = 2^k$, for some integer k, then there exists a set of M orthonormal vectors, each of the form $y^k = \frac{1}{\sqrt{M}} (y^k_1, y^k_2, \ldots y^k_M)$ where y^k_ℓ, $\ell = 1, 2, \ldots M$ is ± 1. In this case, the hypothesis gives $y^k C y^{k'} \leq \frac{c^2}{M}$ and $\mathrm{Tr}\, A \leq c^2$. In general, if $2^k < M < 2^{k+1}$, write $M = \sum_{\nu=0}^{k} \theta_\nu 2^\nu$ where $\theta_k = 1$ and $\theta_\nu = 0$ or 1, $\nu = 0, 1, 2, \ldots k-1$. Let ℓ be the number of non-zero θ_ν's. An $M \times M$ orthogonal matrix can be constructed with ℓ non-zero blocks down the diagonal, the n-th block of dimension $2^{\nu_m} \times 2^{\nu_m}$, $m = 1, 2, \ldots \ell$, where ν_m corresponds to the m-th non-zero θ_ν. In the m-th block place an orthogonal matrix with rows of the form $\frac{1}{\sqrt{2^{\nu_m}}} (u_1, u_2, \ldots u_{2^{\nu_m}})$ where $u_r = \pm 1$. Now let $y^1, y^2, \ldots y^M$ be the rows of this matrix. If part of y^k is contained in the m-th block, $y^k A y^{k'} \leq \frac{c^2}{2^{\nu_m}}$, and there are 2^{ν_m} such y^k's. Hence

$$\mathrm{Tr}\, A \leq \sum_{m=1}^{\ell} 2^{\nu_m} \cdot \frac{c^2}{2^{\nu_m}} = \ell\, c^2 \leq k\, c^2 \leq (\log_2 M)\, c^2.$$

Appendix B.

The Characteristic function of log $\underset{\sim}{\lambda}_3$

Let X_ν, $\nu = 1, 2, \ldots M$ be independent with the density of X_ν given by

$$f_\nu(X) = \frac{(1-b_\nu)^n}{\Gamma(n)} e^{-(1-b_\nu)X}, \quad X > 0 \quad |b_\nu| < 1, \ \nu=1,2,\ldots M,$$

$$= 0 \qquad \text{otherwise}$$

and let

$$\lambda_3 = \frac{\prod_{\nu=1}^{M} X_\nu}{\left(\frac{1}{M}\sum_{\nu=1}^{M} X_\nu\right)^M}$$

$$\Phi_3(s) = E\, e^{is \log \lambda_3} = E(\lambda_3)^{is}$$

$$= M^{isM} \prod_{\nu=1}^{M} (1-b_\nu)^n \Gamma(n)^{-M} \int_0^\infty \ldots \int_0^\infty (\prod_\nu X_\nu)^{is+n-1} (\sum_\nu X_\nu)^{-isM} e^{-\sum_\nu X_\nu + \sum_\nu b_\nu X_\nu} \qquad (13.1)$$

Making the change of variables $Y_\nu = \sum_{j=1}^{\nu} X_\nu$, $\nu = 1, 2, \ldots, M$ and $Z_\nu = Y_\nu / Y_{\nu+1}$, $\nu = 1, 2, \ldots, M-1$, $Z_M = Y_M$ gives

$$X_\nu = (1-Z_{\nu-1}) \prod_{\ell=\nu}^{M} Z_\ell, \quad Z_0 = 0, \; \nu = 1, 2, \ldots, M$$

$$\Pi dX_\nu = \prod_{\nu=1}^{M} Z_\nu^{\nu-1} dZ_\nu .$$

and

$$\Phi_3(s) = M^{isM} \prod_{\mu=1}^{M} (1-b_\mu)^n \Gamma(n)^{-M} \{ \int_0^\infty Z_M^{Mn-1} e^{-Z_M} dZ_M \tag{B.2}$$

$$\times \prod_{\nu=1}^{M-1} \int_0^1 dZ_\nu \cdot Z_\nu^{(is+n)\nu-1} (1-Z_\nu)^{is+n-1}$$

$$\times e^{-\sum_{\nu=1}^{M} b_\nu (1-Z_{\nu-1}) \prod_{\ell=\nu}^{M} Z_\ell} \}.$$

Expand the second exponential in a power series, the k-th term of which is

$$\frac{1}{k!} \left(\sum_{\nu=1}^{M} b_\nu (1-Z_{\nu-1}) \prod_{\ell=\nu}^{M} Z_\ell \right)^k$$

The coefficient of $b_1^{q_1} b_2^{q_2} \ldots b_M^{q_M}$, $\sum_{\nu=1}^{M} q_\nu = k$ in the expansion of this term is given by

$$\frac{1}{q_1! \ldots q_M!} \prod_{\nu=1}^{M} Z_\nu^{\sum_{\ell=1}^{\nu} q_\ell} \prod_{\nu=1}^{M-1} (1-Z_\nu)^{q_{\nu+1}}, \tag{B.3}$$

and the coefficient of $b_1^{q_1} b_2^{q_2} \ldots b_M^{q_M}$, $\sum_{\nu=1}^{M} q_\nu = k$ in the expansion of the term in brackets in (B.2) is

$$\int_0^\infty dZ_M \cdot Z_M^{Mn-1+k} \prod_{\nu=1}^{M-1} \int_0^1 Z_\nu^{(is+n)\nu-1+\sum_{j=1}^{\nu} q_j} (1-Z_\nu)^{is+n-1+q_{\nu+1}} dZ_\nu \qquad \text{(B.4)}$$

Now, for $|b_\nu| < 1$, $= 1, 2, \ldots M$, $|t| < 1$, we have

$$\prod_{\nu=1}^{M} (1-b_\nu t)^{-(is+n)} = \sum_{k=0}^{\infty} t^k \sum_{\pi_k} \prod_{\nu=1}^{M} \binom{is+n-1+q_\nu}{q_\nu} b_\nu^{q_\nu} \quad . \qquad \text{(B.5)}$$

where $\sum_{\pi_k}$ is the sum over all partitions $(q_1, q_2, \ldots q_M)$ of k such that $\sum_{\nu=1}^{M} q_\nu = k$. It follows from (B.4) and (B.7) with s = 0, that we may write

$$\Phi_3(s) = M^{isM} \prod_{u=1}^{M} (1-b_u)^n \sum_{k=0} \sum_{\pi_k} \Phi_{q_1, q_2, \ldots q_M}(s) \prod_{\nu=1}^{M} \binom{n-1+q_\nu}{q_\nu} b_\nu^{q_\nu}$$

(B.6)

where $\Phi_{q_1, q_2, \ldots q_M}(s)$ is given by

$$\Phi_{q_1, q_2, \ldots q_M}(s) = \prod_{\nu=1}^{M-1} E\, e^{is\log Z_\nu^{\nu}(1-Z_\nu)}$$

with Z_ν distributed as $\beta_{\xi,\eta}$, with $\xi = \nu n + \sum_{j=1}^{\nu} q_\nu$, $\eta = n + q_{\nu+1}$.

Performing the integration in (B.4) we find that (B.4), the coefficient of $b_1^{q_1} b_2^{q_2} \ldots b_M^{q_M}$, $\sum_{\nu=1}^{M} q_\nu = M$ is

$$\frac{\Gamma(Mn+k)}{\prod_{\nu=1}^{M} q_\nu!} \prod_{\nu=1}^{M-1} \frac{\Gamma((is+n)\nu + \sum_{j=1}^{\nu} q_j)\,\Gamma(is+n+q_{\nu+1})}{\Gamma((is+n)\nu+1+\sum_{j=1}^{\nu+1} q_j)} \tag{B.7}$$

$$= \frac{(Mn+k)}{\prod_{\nu=1}^{M} q_\nu!} \; \frac{\prod_{\nu=1}^{M} \Gamma(is+n+q_\nu)}{\Gamma((is+n)M+k)} \tag{B.8}$$

$\Phi_o(s)$, the characteristic function of λ_3 under M_3 is, from (B.2) and (B.10) with $|b_\nu| = 0$,

$$\Phi_o(s) = M^{isM} \frac{\Gamma(is+n)^M}{\Gamma(n)^M} \cdot \frac{\Gamma(Mn)}{\Gamma(M(is+n))} \tag{B.9}$$

By using (B.8), the coefficient of $b_1^{q_1}\, b_2^{q_2} \ldots b_M^{q_M}$ in $\Phi_3(s)$, $\sum_{\nu=1}^{M} q_\nu = k$, may be written

$$\prod_{u=1}^{M} \Phi_o(s)\ (1-b_u)^n\ \psi_k(s) \prod_{\nu=1}^{M} \binom{is+n-1+q_\nu}{q_\nu} \tag{B.10}$$

where

$$\Psi_k(s) = \frac{\Gamma((is+n)M)}{\Gamma(Mn)} \frac{\Gamma(Mn+k)}{\Gamma((is+n)M+k)} \tag{B.11}$$

We have, using (B.5)

$$\Psi(s) = \Phi_o(s) \prod_{u=1}^{M} (1-b_u)^n \sum_{k=0}^{\infty} \Psi_k(s) \; \theta_k(s) \tag{B.12}$$

where $\theta_k(s)$ is defined by

$$\sum_{k=0}^{\infty} \theta_k(s) t^k = 0 \prod_{\nu=1}^{M} (1-b_\nu t)^{-(is+n)} \tag{B.13}$$

REFERENCES

[1] Anderson, T. W. (1958). An Introduction to Multivariate Analysis. John Wiley and Sons: New York.

[2] Bateman Manuscript Project (1954). Tables of Integral Transforms, Vol. 1. McGraw-Hill: New York, p. 232.

[3] Consul, P. C. (1965). The exact distribution of certain likelihood criteria useful in multivariate analysis. Acad. Roy. Belg. Bull. Cl. Sci. 51, 683-691.

[4] Goodman, N. R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution. Ann. Math. Statist. 34, 152-177.

[5] Grenander, U. and Szego, G., (1958). Toeplitz Forms and Their Applications. University of California Press, Berkeley.

[6] Khatri, C. G. (1965). Classical statistical analysis based on a certain multivariate complex distribution. Ann. Math. Statist. 36, 98-114.

[7] Koopmans, L. H. (1965). Multivariate analysis of stochastic processes. Ann. Math. Statist. 35, 1765-1778.

[8] Wald, A., and Brookner, R. J. (1941). On the distribution of Wilks' statistic for testing the independence of several groups of variables. Ann. Math. Statist. 12, 137-152.

[9] Whittaker, E. T., and Watson, G. N. (1927). A Course of Modern Analysis, ed. 4, Section 12. Cambridge University Press: Cambridge.

[10] Whittle, P. (1952). Tests of fit in time series. Biometrika 39, 309-318.

[11] Wilks, S. S. (1938). The sampling distribution of the criterion λ_H when the hypothesis tested is not true. Biometrika 29, 124-132.