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UNIFORM CONVERGENCE RATES FOR CERTAIN APPROXIMATE SOLUTIONS TO FIRST KIND INTEGRAL EQUATIONS

by

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ABSTRACT

This note considers convergence rates for certain approximate solutions to Fredholm integral equations of the first kind, namely

$$u(t) = \int_S K(t,s)\, z(s)\, ds, \qquad t \in T \qquad (*)$$

where S, T are closed, bounded intervals of the real line, and $K(t,s)$ is a given kernel on $T \times S$ with appropriate properties. It is desired to approximate $z(s)$ given $u(t)$, for $t = t_1, t_2, \ldots t_n$. We assume that z is an element of a Reproducing Kernel Hilbert space $\mathcal{H}_R$ with reproducing kernel $R(s,s')$, and we let the approximate solution to $(*)$ be that element $z \in \mathcal{H}_R$ which minimizes $\|z\|_R^2$ subject to

$$u(t_i) = \int_S K(t_i,s)\, z(s)\, ds, \qquad i = 1, 2, \ldots n$$

Let $\|\Delta\| = \sup_i |t_{i+1} - t_i|$, where t_1 and t_n are the boundaries of T, and let $Q(t,t')$ be defined by

$$Q(t,t') = \int_S \int_S K(t,s)\, R(s,s')\, K(t',s')\; ds\; ds'$$

Loosely speaking, we show that, if Q has smoothness properties similar to those of the Green's function of a $2m$ th order differential operator, and z is sufficiently smooth to insure that u given by $(*)$ has a representation of the form

$$u(t) = \int_T Q(t,t')\, \rho(t')\, dt'$$

for some bounded ρ, then if the approximate solution to $(*)$ above be denoted by $P_{V_n} z$, we have

$$|z(s) - (P_{V_n} z)(s)| = 0\,(\|\Delta\|^m) \qquad s \in S$$

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1. Introduction: This note continues the study, begun in [3] of methods of solution of Fredholm Integral Equations of the first kind, namely

$$u(t) = \int_S K(t,s)\; z(s)\, ds \qquad , t \in T \tag{1.1}$$

where S, T are closed, bounded intervals of the real line, $K(t,s)$ is a given kernel on $T \times S$ with appropriate properties, and $u(t)$ is given for $t \in \Delta$; $\Delta = \{t_1 < t_2 < \ldots < t_n,\ t_i \in T\}$.

In [3], it was assumed that $z \in \mathcal{H}_R$, where $\mathcal{H}_R$ is the reproducing kernel Hilbert Space with reproducing kernel $R(s,s')$. $R(s,s')$ is a given positive definite kernel on $S \times S$. Let

$$Q(t,t') = \int_S \int_S K(t,s)\, R(s,s')\, K(t',s')\; ds ds'. \tag{1.2}$$

It was shown in [3] that, if $Q(t,t)$ is well defined and finite as a Rieman integral for every $t \in T$, then the linear functional Λ_t on $\mathcal{H}_R$, defined, for fixed $t \in T$ by

$$\Lambda_t z = \int_S K(t,s)\, z(s)\, ds \tag{1.3}$$

is continuous on $\mathcal{H}_R$ and has the representation

$$\Lambda_t z = \langle \eta_t, z \rangle_R , \quad t \in T, \quad z \in \mathcal{H}_R, \tag{1.4}$$

where $\eta_t \in \mathcal{H}_R$ is defined by

$$\eta_t(s) = \int_S K(t,u)\, R(s,u)\, du \tag{1.5}$$

and $\langle \cdot, \cdot \rangle_R$ denotes inner product in $\mathcal{H}_R$. We will denote the subspace of $\mathcal{H}_R$ spanned by $\{\eta_{t_i}, t_i \in \Delta\}$ by V_n and the projection operator in $\mathcal{H}_R$ onto V_n by P_{V_n}.

Suppose $u(t_i)$ and $\eta_{t_i}(s)$ are known exactly for $t_i \in \Delta$, $s \in S$. Then it was proposed in [3] to choose as a "solution" to

$$u(t_i) = \int_S K(t_i, s)\, z(s)\, ds \quad , \quad t_i \in \Delta \tag{1.6}$$

that element $z \in \mathcal{H}_R$ which minimizes $\|z\|_R^2$ subject to the constraints (1.6). Since we may write (1.6) as

$$u(t_i) = \langle \eta_{t_i}, z \rangle_R \quad , \quad t_i \in \Delta , \tag{1.7}$$

the solution is $P_{V_n} z$, and is explicitly given, in the case $\dim V_n = n$ by

$$(P_{V_n} z)(s) = (\eta_{t_1}(s), \eta_{t_2}(s), \ldots \eta_{t_n}(s)\, \bar{Q}^{-1} (u(t_1), u(t_2), \ldots u(t_n))', \tag{1.8}$$

where $\bar{Q}$ is the $n \times n$ Grammian of $\{\eta_{t_i}\}_{i=1}^{n}$, with i, j^{th} entry given (see [3]) by

$$\langle \eta_{t_i}, \eta_{t_j} \rangle_R = Q(t_i, t_j) \tag{1.9}$$

Let $\|\Delta\| = \sup_i |t_{i+1} - t_i|$, where, without loss of generality, we let t_1 and t_n be the boundaries of T.

The purpose of this note is to investigate the rate of convergence of $(P_{V_n} z)(s)$ to $z(s)$, as $\|\Delta\| \to 0$. Let $R_s \in \mathcal{H}_R$ be that element of $\mathcal{H}_R$ whose value at s' is given by

$$R_s(s') = R(s, s') \tag{1.10}$$

By the properties of reproducing kernel spaces,

$$\langle R_s, z \rangle_R = z(s) \ , \quad s \in S, \ z \in \mathcal{H}_R \tag{1.11}$$

and hence

$$\begin{aligned} |z(s) - (P_{V_n} z)(s)| &= |\langle z - P_{V_n} z, R_s - P_{V_n} R_s \rangle_R| \\ &\leq \|z - P_{V_n} z\|_R \ \|R_s - P_{V_n} R_s\|_R \\ &\leq \|z - P_{V_n} z\|_R \ \|R_s\|_R \\ &= \|z - P_{V_n} z\|_R \ R^{\frac{1}{2}}(s,s) \end{aligned} \tag{1.12}$$

Suppose $z \in \mathcal{H}_R$ and

$$0 \equiv \int_S K(t,s)\, z(s)\, ds, \quad t \in T, \tag{1.13}$$

entails that $z(s) \equiv 0$.

Equation (1.13) may be written

$$0 = \langle \eta_t, z \rangle_R \ , \quad t \in T \implies z = 0, \tag{1.14}$$

thus, in this case, $\{ \eta_t, t \in T \}$ span $\mathcal{H}_R$. $R(s,s')$ continuous on $S \times S$ insures that $\mathcal{H}_R$ is separable, and, as $\|\Delta\| \to 0$, we must have $\|z - P_{V_n} z\|_R \to 0$, any fixed z, (including $z = R_s$). However it appears that no rate holding uniformly for $z \in \mathcal{H}_R$ can be found. In this note we exhibit convergence rates for $\| z - P_{V_n} z \|_R$ when z belongs to a special subset of $\mathcal{H}_R$. More specifically, the main purpose of this paper is to prove

Theorem 4.

Let $z \in \mathcal{H}_R$, have a representation

$$z(s) = \int_T \eta_t(s) \rho(t)\, dt \tag{1.15}$$

for some bounded ρ, and suppose $\{\eta_t, t \in T\}$ span $\mathcal{H}_R$. Let $Q(t,t')$ satisfy

(i) $\frac{\partial^\ell}{\partial t^\ell} Q(t,t')$ exists and is continuous on $T \times T$ for $t \neq t'$, $\ell = 0,1,2,\ldots 2m$, $\frac{\partial^\ell}{\partial t^\ell} Q(t,t')$ exists and is continuous on $T \times T$ for $\ell = 0,1,2,\ldots 2m-2$

ii) $\lim_{t \uparrow t'} \frac{\partial^{2m-1}}{\partial t^{2m-1}} Q(t,t')$ and $\lim_{t \downarrow t'} \frac{\partial^{2m-1}}{\partial t^{2m-1}} Q(t,t')$

exist and are bounded for all $t' \in T$.

Then

$$\| z - P_{V_n} z \|_R = O(\| \Delta \|)^m$$

The proof, though long, rests only on an isomorphism between two appropriately chosen Hilbert spaces and the Newton form of the remainder for Lagrange Interpolation. Section 2 is devoted to the proof, which proceeds via three preliminary theorems. Section 3 is devoted to a simple example and discussion.

In the example of Section 3, $z \in \mathcal{H}_R$ implies that $z^{(k)} \in L_2(S)$ and u of 1.1 satisfies $u^{(m)} \in L_2(T)$, where $k + \ell = m$ and $K(t,s)$ is a Green's function for an ℓ th order differential operator. In that example, which is apparently typical of the general situation, the restriction (1.15) implies that $u^{(m+j)}$, $j = 0, 1, 2, \ldots m-1$ is continuous and $u^{(2m)}$ is bounded. Thus, loosely speaking, our results apply to the situation in which u is slightly more than twice as smooth as in implied by the assumption $z \in \mathcal{H}_R$.

Theorem 4 has an interesting

Corollary Let N be a continuous linear functional on $\mathcal{H}_R$ with representer δ,

$$Nz = \langle \delta, z \rangle_R \quad , \quad z \in \mathcal{H}_R \tag{1.16}$$

and suppose that δ has the form

$$\delta(s) = \int_T \eta_t(s)\, \psi(t)\, dt \tag{1.17}$$

for some bounded ψ. Let Nz be approximated by $N(P_{V_n} z)$. Then

$$|Nz - N(P_{V_n} z)| = O(\|\Delta\|^m) . \quad , z \in \mathcal{H}_R. \tag{1.18}$$

This is an immediate consequence of Theorem 4 upon noting that

$$|Nz - N(P_{V_n} z)| = |\langle z - P_{V_n} z, \delta - P_{V_n} \delta \rangle_R|$$

$$\leq \|z - P_{V_n} z\|_R \, \|\delta - P_{V_n} \delta\|_R \tag{1.19}$$

and

$$\|\delta - P_{V_n} \delta\|_R = O(\|\Delta\|^m) \tag{1.20}$$

2. Proof of Theorem 4

Theorem 4 is an immediate consequence of Theorems 1, 2, and 3 below. Theorem 1 is of independent interest and is discussed in Section 3.

Theorem 1.

Let $Q(t,t')$ be a positive definite kernel on $T \times T$ and let $\mathcal{H}_Q$ be the reproducing kernel Hilbert space with Q as reproducing kernel.

Suppose

i) $\frac{\partial^\ell}{\partial t^\ell} Q(t,t')$ exists and is continuous on $T \times T$ for (2.1)

$t \neq t'$, $\ell = 0,1,2,\ldots 2m$; $\frac{\partial^\ell}{\partial t^\ell} Q(t,t')$ exists and is continuous on $T \times T$ for $\ell \leq 2m-2$.

$$\text{ii)} \quad \lim_{t \uparrow t'} \frac{\partial^{2m-1}}{\partial t^{2m-1}} Q(t,t') \quad \text{and} \quad \lim_{t \downarrow t'} \frac{\partial^{2m-1}}{\partial t^{2m-1}} Q(t,t') \tag{2.2}$$

exist and are bounded for all $t' \in T$. Let $\mathcal{H}_{T_n}$ be the subspace of $\mathcal{H}_Q$ spanned by the elements $\{Q_{t_i}\}_{i=1}^n$ where $\{t_i\}_{i=1}^n$ are n distinct points in T and Q_{t_i} is that element of $\mathcal{H}_Q$ whose value at t is given by

$$Q_{t_i}(t) = Q(t_i, t) \qquad i = 1,2,\ldots n \tag{2.3}$$

Let P_{T_n} be the projection operator onto $\mathcal{H}_{T_n}$ and $\langle \cdot , \cdot \rangle_Q$ denote inner product in $\mathcal{H}_Q$.

Let $u \in \mathcal{H}_Q$ have a representation of the form

$$u(t) = \int_T Q(t,t')\,\rho(t')\,dt' \tag{2.4}$$

for some continuous ρ. Without loss of generality let t_1 and t_n be the boundary points of T, and let $n = N(2m-1)$ for some integer N. Let I_k be the interval

$$I_k = [t_{(k+1)(2m-1)+1} - t_{k(2m-1)+1}] \quad , \quad k=0,1,2,\ldots N-1.$$

Then

$$\|u - P_{T_n} u\|_Q^2 \leq \sup_t \rho^2(t)\,(2m+1)\,(t_n - t_1) \quad \times$$

$$\left[\sup_{\xi \neq t} \left| \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, t) \right| + 2 \sup_k \sup_{\xi, t \in I_k} \left| \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, t) \right| \right] \times$$

$$\sum_{k=0}^{N-1} \left(t_{(k+1)(2m-1)+1} - t_{k(2m-1)+1} \right)^{2m+1} \tag{2.5}$$

$$= O(\|\Delta\|^{2m})$$

Remark: Since $u(t_i) = \langle Q_{t_i}, u \rangle_Q$, $i = 1, 2, \ldots n$, $P_{T_n} u$ is that element of minimum $\mathcal{H}_Q$ norm which interpolates to $u(t_i)$, $i = 1, 2, \ldots n$, $\langle P_{T_n} u, Q_{t_i} \rangle_Q = (P_{T_n} u)(t_i) = u(t_i)$.

Proof:

If $\tilde{u}$ is any element in $\mathcal{H}_Q$ of the form

$$\tilde{u} = \sum Q_{t_i} \int_0^1 c_i(t)\, \rho(t)\, dt \tag{2.6}$$

then, since $\tilde{u} \in \mathcal{H}_{T_n}$ we have

$$\| u - P_{T_n} u \|_Q^2 \leq \| u - \tilde{u} \|_Q^2 \tag{2.7}$$

The proof proceeds by finding a set of functions $\{c_i(t)\}_{i=1}^n$ so that $\|u - \tilde{u}\|_Q^2$ with $\tilde{u}$ defined by (2.6) is bounded by the right hand side of (2.5).

Now, since u satisfying (2.4) satisfies

$$\langle u,u\rangle_Q = \int_0^1\int_0^1 \rho(t)\, Q(t,t')\,\rho\,(t')\,dt\,dt' \tag{2.8}$$

$$\langle u,\, Q_{t_i}\rangle_Q = u(t_i) = \int_0^1 Q_{t_i}(t)\,\rho(t)\,dt \tag{2.9}$$

it follows that

$$\| u - \tilde{u} \|_Q^2 = \int_0^1\int_0^1 \rho(t)\rho(t') \left\langle Q_t - \sum_{i=1}^m c_i(t)\, Q_{t_i} \,,\, Q_{t'} - \sum_{j=1}^m c_j(t')Q_{t_j}\right\rangle_Q dt\,dt' \tag{2.10}$$

where Q_t is, for each fixed t, that element in $\mathcal{H}_Q$ whose value at t' is given by $Q_t(t') = Q(t,t')$. For $t \in I_k$ we will approximate Q_t by that linear combination of $\left\{Q_{t_{k(2m-1)+\ell}}\right\}_{\ell=1}^{2m}$ which corresponds to Lagrange (polynomial) interpolation of degree 2m-1. More precisely, let

$$p_{k,\ell}(t) = \frac{\prod\limits_{\substack{j=1\\ j\neq \ell}}^{2m} (t - t_{k(2m-1)+j})}{\prod\limits_{\substack{j=1\\ j\neq \ell}}^{2m} (t_{k(2m-1)+\ell} - t_{k(2m-1)+j})} \qquad t \in I_k \tag{2.11}$$

$$= 0 \qquad t \notin I_k$$

$$k = 0,1,2,\ldots N-1$$

$$\ell = 1,2,\ldots\; 2m$$

and let

$$c_{k(2m-1)+\ell}(t) = p_{k,\ell}(t) \quad , \quad k = 0,1,2,\ldots N-1 \tag{2.12}$$

$$\ell = 1,2,\ldots 2m-2$$

$$\text{and} \quad (k,\ell) = (0,1)$$

$$= p_{k-1,1}(t) + p_{k,1}(t) \quad , \quad k = 1,2,\ldots n-1$$

Then, for $f \in \mathcal{H}_Q$, $t \in I_k$,

$$\left\langle Q_t - \sum_{i=1}^{n} c_i(t)\, Q_{t_i}, f \right\rangle_Q = \left\langle Q_t - \sum_{\ell=1}^{2m} p_{k,\ell}(t)\, Q_{t_{k(2m-1)+\ell}}, f \right\rangle_Q$$

$$= f(t) - \sum_{\ell=1}^{2m} p_{k,\ell}(t)\, f(t_{k(2m-1)+\ell}) \qquad (2.13)$$

where $\sum_{\ell=1}^{2m} p_{k,\ell}(t)\, f(t_{k(2m-1)+\ell})$ is, for $t \in I_k$, the 2m-1 st degree polynomial interpolating to f at $t = t_{k(2m-1)+1}, t_{k(2m-1)+2}, \ldots t_{(k+1)(2m-1)+1}$. It remains only to show that $\tilde{u}$ defined by (2.6) with $\{c_i(t)\}_{i=1}^{n}$ given by (2.12) has the required properties.

By the Newton form of the remainder for Lagrange interpolation, we know (see for example, Isaacson and Keller [2], p. 248) that, for $t \in I_k$

$$f(t) - \sum_{j=1}^{2m} p_{k,j}(t)\, f(t_{k(2m-1)+j})$$

$$= \prod_{j=1}^{2m} (t - t_{k(2m-1)+j})\, f\,[t_{k(2m-1)+1}, t_{k(2m-1)+2}, \ldots t_{(k+1)(2m-1)+1}, t]$$

(2.14)

where $f[t_{k(2m-1)+1}, t_{k(2m-1)+2}, \ldots t_{(k+1)(2m-1)+1}, t]$ is the 2m th order divided difference. Furthermore we know that if f has 2m continuous derivatives in I_k, then

$$f\,[t_{k(2m-1)+1}, t_{k(2m-1)+2}, \ldots t_{(k+1)(2m-1)+1}, t] = \frac{f^{(2m)}}{(2m)!}(\xi), \qquad (2.15)$$

for some $\xi \in I_k$. If we only know that $f^{(2m-1)}(t)$ is continuous except for a finite number of finite jumps, then we may write the $2m$ th order divided difference as a divided difference of two $2m-1$ st order divided differences,

$$f[t_{k(2m-1)+1}, t_{k(2m-1)+2}, \dots t_{(k+1)(2m-1)+1}, t] =$$

$$\frac{1}{(t_{(k+1)(2m-1)+1}-t_{k(2m-1)+1})} \{f[t_{k(2m-1)+1}, t_{k(2m-1)+2}, \dots t_{k(2m-1)+2m-1}, t]-$$

$$f[t_{k(2m-1)+2}, \dots t_{(k+1)(2m-1)+1}, t]\} \qquad (2.16)$$

and know that the term in brackets in (2.16) is bounded in absolute value by $2 \sup_{t \in I_k} |f^{(2m-1)}(t)|$. Now let $f_{t'}$ be that element of $\mathcal{H}_Q$ defined by

$$f_{t'} = Q_{t'} - \sum_{j=1}^{n} c_j(t')\, Q_{t_j} \quad , \qquad (2.17)$$

that is,

$$f_{t'} = Q_{t'} - \sum_{j=1}^{2m} p_{\ell,j}(t')\, Q_{t_{\ell(2m-1)+j}} \qquad \text{for} \quad t' \in I_\ell \qquad (2.18)$$

By assumptions (i) and (ii), for fixed $t' \in I_\ell$, $f_{t'}^{(2m)}(t)$ is continuous for $t \notin I_\ell$, and, for $t \in I_\ell$, $f_{t'}^{(2m-1)}(t)$ has bounded left and right derivatives. Thus, for $k \neq \ell$ and $\rho = \sup_t |\rho(t)|$, we have

$$\left| \int_{I_k}\int_{I_\ell} \rho(t)\,\rho(t')\left\langle Q_t - \sum_{j=1}^{2m} p_{k,j}(t)\, Q_{t_{k(2m-1)+j}}\,,\; Q_{t'} - \sum_{j=1}^{2m} p_{\ell,j}(t')Q_{t_{\ell(2m-1)+j}} \right\rangle_Q dt\,dt'\right.$$

$$\leq \rho^2 \int_{I_k}\int_{I_\ell} \left| \left\langle Q_t - \sum_{j=1}^{2m} p_{k,j}(t)\, Q_{t_{k(2m-1)+j}},\; f_{t'} \right\rangle_Q \right| dt\,dt'$$

$$= \rho^2 \int_{I_k}\int_{I_\ell} \left| \prod_{j=1}^{2m} (t-t_{k(2m-1)+j})\, f_{t'}\,[t_{k(2m-1)+1},\; t_{k(2m-1)+2}, \ldots t_{(k+1)(2m-1)+1}, t] \right| dt\,dt'$$

$$\leq \rho^2\, (t_{(\ell+1)(2m-1)+1} - t_{\ell(2m-1)+1})\,(t_{(k+1)(2m-1)+1} - t_{k(2m-1)+1})^{2m+1}$$

$$\times\ \left| f_{t'}^{(2m)}(\xi) \right| \qquad (2.19)$$

for some $\xi \in I_k$. Since $|p_{\ell,j}(t)| \leq 1$, we have, for $k \neq \ell$ that

$$|f_{t'}^{(2m)}(\xi) \leq (2m+1) \sup_{\substack{\xi \in I_k \\ t' \in I_\ell}} \left| \frac{\partial^{2m}}{\partial \xi^{2m}}\, Q(\xi, t') \right| . \qquad (2.20)$$

For $k=\ell$, by use of (2.14) and (2.16), we have

$$\left| \int_{I_k}\int_{I_k} \rho(t)\,\rho(t') \left\langle Q_t - \sum_{j=1}^{2m} p_{k,j}(t)\, Q_{t_{k(2m-1)+j}}\,,\; f_{t'} \right\rangle_Q dt\,dt' \right|$$

$$\leq \rho^2 \int_{I_k}\int_{I_k} \frac{\left|\prod_{j=1}^{2m} (t-t_{k(2m-1)+j})\right|}{(t_{(k+1)(2m-1)+1} - t_{k(2m-1)+1})} \quad \times$$

$$\left| f_{t'}\left[t_{k(2m-1)+1}, \ldots t_{k(2m-1)+2m-1}, t\right] - f_{t'}\left[t_{k(2m-1)+2}, \ldots t_{(k+1)(2m-1)+1}, t\right] \right| dt\,dt'$$

$$\leq \rho^2\, (t_{(k+1)(2m-1)+1} - t_{k(2m-1)+1})^{2m+1}\; 2(2m+1) \sup_{\xi, t' \in I_k} \left| \frac{\partial^{2m-1}}{\partial \xi^{2m-1}}\, Q(\xi, t') \right| \qquad (2.21)$$

Putting together (2.10), (2.18), (2.19), (2.20) and (2.21) gives

$$\|u - \tilde{u}\|_Q^2$$

$$\leq \sum_{k,\ell=0}^{N-1} \left| \int_{I_k}\int_{I_\ell} \rho(t)\rho(t') \left\langle Q_t - \sum_{j=1}^{2m} p_{k,j}(t) Q_{t_{k(2m-1)+j}}, Q_{t'} - \sum_{j=1}^{2m} p_{\ell,j}(t') Q_{t_{\ell(2m-1)+j}} \right\rangle_Q dt\, dt' \right|$$

$$\leq \rho^2 (2m+1) \left[\sup_{\xi \neq t} \left| \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, t) \right| + 2 \sup_k \sup_{\xi, t \in I_k} \left| \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, t) \right| \right].$$

$$\sum_{k=0}^{N-1} (t_{(k+1)(2m-1)+1} - t_{k(2m-1)+1})^{2m+1} \qquad (2.22)$$

$$= O(\|\Delta\|^{2m})$$

Theorem 2.

Let $R(s,s')$ be a positive definite kernel on $S \times S$ with associated reproducing kernel Hilbert space $\mathcal{H}_R$, let $\eta(t,s)$ defined on $T \times S$ have the property that for each fixed t the function of s defined by $\eta_t(s)$, where

$$\eta_t(s) = \eta(t,s) \qquad (2.23)$$

satisfies $\eta_t \in \mathcal{H}_R$. Let $Q(t,t')$ be the positive definite kernel on $T \times T$ defined by

$$Q(t,t') = \langle \eta_t, \eta_{t'} \rangle_R \quad , \quad t, t' \in T \qquad (2.24)$$

and let $\mathcal{H}_Q$ be the reproducing kernel Hilbert space with Q as reproducing kernel.

Let V be the subspace of $\mathcal{H}_R$ spanned by the family $\{\eta_t, t \in T\}$, and let V_n be the subspace of V spanned by $\{\eta_{t_i}, t_i \in \Delta\}$. Let P_V and P_{V_n}

be the projection operators in $\mathcal{H}_R$ onto V and V_n respectively. Then

$$\| P_V z - P_{V_n} z \|_R^2 = \| u - P_{T_n} u \|_Q^2 \tag{2.25}$$

where $u \in \mathcal{H}_Q$ is defined by

$$u(t) = \langle \eta_t, z \rangle_R \tag{2.26}$$

and P_{T_n} is the projection operator in $\mathcal{H}_Q$ onto the subspace $\mathcal{H}_{T_n}$ spanned by the elements $\{Q_{t_i}, t_i \in \Delta\}$.

Proof:

Since

$$\langle Q_t, Q_{t'} \rangle_Q = Q(t,t') = \langle \eta_t, \eta_{t'} \rangle_R, \qquad t, t' \in T, \tag{2.27}$$

there is an isometric isomorphism between $\mathcal{H}_Q$ and V generated by the correspondence " $\sim$ ",

$$Q_t \in \mathcal{H}_Q \sim \eta_t \in V \qquad t \in T \tag{2.28}$$

Obviously

$$\mathcal{H}_{T_n} \sim V_n$$

under this is isomorphism. Furthermore, since for $z \in \mathcal{H}_R$,

$$\langle \eta_t, P_V z \rangle_R = \langle \eta_t, z \rangle_R = u(t) = \langle Q_t, u \rangle_Q, \tag{2.29}$$

we have

$$P_V z \sim u,$$

and

$$P_{V_n} z \sim P_{T_n} u \tag{2.30}$$

and hence

$$\| P_V z - P_{V_n} z \|_R^2 = \| u - P_{T_n} u \|_Q^2 \tag{2.31}$$

Theorem 3.

Let ρ be bounded on T and suppose z has a representation

$$z(s) = \int_T \eta(t,s)\, \rho(t)\, dt \tag{2.32}$$

where $\eta(t,s)$ satisfies the hypotheses of Theorem 2.

Then $z \in V$ and $z \sim u$ under the correspondence "$\sim$" of (2.18) where

$$u(t) = \int_T Q(t,t')\, \rho(t')\, dt' \tag{2.33}$$

Proof:

Let $\Pi_\ell = \{t_{11}, t_{12}, \ldots t_{1\ell}\}$, $\ell = 1, 2, \ldots$ be, for each ℓ, a partition of T, such that, for every t, the Riemann sums for Π_ℓ for the integral

$$\int_T Q(t,t')\, \rho(t')\, dt' \tag{2.34}$$

converge.

Then $z_{(\ell)}$, $\ell = 1, 2, \ldots$ defined by

$$z_{(\ell)}(s) = \sum_{j=1}^{\ell-1} \eta_{t_{j\ell}}(s)\, \rho(t_{j\ell})(t_{j+1,\ell} - t_{j\ell}) \quad , \ell = 1, 2, \ldots \tag{2.35}$$

is a Cauchy sequence of elements in V which converge pointwise to $z(s)$ of (2.32) and $u_{(\ell)}$, $\ell = 1, 2, \ldots$ defined by

$$u_{(\ell)}(t) = \left\langle \eta_t, z_{(\ell)} \right\rangle_R = \left\langle \eta_t, \sum_{j=1}^{\ell-1} \eta_{t_{j\ell}}\, \rho(t_{j\ell})(t_{j+1,\ell} - t_{j,\ell}) \right\rangle_R \quad \ell = 1, 2, \ldots \tag{2.36}$$

is a Cauchy sequence of elements in $\mathcal{H}_Q$ which converge pointwise to u(t) given by (2.33). But, by (2.30), $u_{(\ell)} \sim z_{(\ell)}$ so we must have $u \sim z$ with u and z defined by (2.32) and (2.33).

Theorem 4

Let

$$u(t) = \int_S K(t,s)\, z(s) ds \qquad (2.37)$$

where z is assumed to be in $\mathcal{H}_R$, the reproducing kernel Hilbert space with reproducing kernel R(s, s'). Let Q(t, t') defined by

$$Q(t,t') = \int_S \int_S K(t,s)\, R(s,s')\, K(t',s')\, ds\, ds' \qquad (2.38)$$

be well defined as a Riemann integral and

satisfy hypotheses (i) and (ii) of Theorem 1. Suppose $z \in \mathcal{H}_R$ and $u(t) \equiv 0 \Rightarrow z = 0$, and suppose z has a representation of the form

$$z(s) = \int_T \eta(t,s)\, \rho(t)\, dt \qquad (2.39$$

where

$$\eta(t,s) = \int_S K(t,s')\, R(s,s')\, ds'. \qquad (2.40)$$

Let V_n be the subspace of $\mathcal{H}_R$ spanned by the elements $\{\eta_{t_i},\ t_i \in \Delta\}$ where η_{t_i} is defined by

$$\eta_{t_i}(s) = \eta(t_i, s) \qquad (2.41)$$

and suppose P_{V_n} is the projection operator onto V_n. Then

$$|z(s) - (P_{V_n} z)(s)| \le R^{\frac{1}{2}}(s,s) \cdot \sup_t \rho^2(t) \cdot (2m+1)$$

$$\left[\sup_{\xi \neq t} \left| \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi,t) \right| + 2 \sup_k \sup_{\xi, t \in I_k} \left| \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi,t) \right| \right] \quad x$$

$$\sum_{k=o}^{N-1} (t_{(k+1)(2m-1)+1} - t_{k(2m-1)+1})^{2m+1}$$

$$= O(\|\Delta\|)^{2m} \tag{2.42}$$

Proof

$$|z(s) - (P_{V_n} z)(s)| = | \langle z - P_{V_n} z, R_s \rangle_R |$$

$$\le \| z - P_{V_n} z \|_R \, \| R_s \|_R \tag{2.43}$$

Since $u(t) = \langle \eta_t, z \rangle_R$, the assumption $z \in \mathcal{H}_R$, $u(t) = 0$, $t \in T \Rightarrow z = 0$ is equivalent to the assertion $\{\eta_t, t \in T\}$ span $\mathcal{H}_R$. Then, by Theorem 3, $z \sim u \in \mathcal{H}_Q$ given by

$$u(t) = \int_T Q(t,t')\, \rho(t')\, dt' \tag{2.44}$$

under the correspondence (2.28) and hence

$$\| z - P_{V_n} z \|_R^2 = \| u - P_{T_n} u \|_Q^2 \tag{2.45}$$

where P_{T_n} is as in Theorem 1. Application of Theorem 1 to the right hand side of (2.45) then gives the result.

3. Discussion

In an attempt to give the reader a feel for what is going on, we discuss a very simple example. There are several interesting points which will become clear from the example. First, assume that $\{\eta_t, t \in T\}$ span $\mathcal{H}_R$ and let K be the 1:1 linear operator from $\mathcal{H}_R$ onto $\mathcal{H}_Q$ defined by

$$(Kz)(t) = \int_S K(t,s)z(s)ds \tag{3.1}$$

Then the "solution" $P_{V_n} z$ is given by

$$P_{V_n} z = K^{-1}(P_{T_n} u) \tag{3.2}$$

where $P_{T_n} u$ is that element of $\mathcal{H}_Q$ of minimum $\mathcal{H}_Q$ norm which interpolates to $u(t_i)$, $i=1,2,\ldots n$. Secondly, we do not so far have the best error rate for $|z(s) - (P_{V_n} z)(s)|$ since we have not discussed the rate at which $\|R_s - P_{V_n} R_s\| \to 0$. This is done in the example presented here. We remark that if $R(s,s')$ and $K(t,s)$ are appropriate Green's functions, then the theorems presented here are intimately related to some convergence theorems in the theory of spline functions. We will illustrate this remark with respect to certain polynomial splines.

Let $S = [0,1]$, $T = [0,1]$ and let

$$R(s,s') = \int_0^1 \frac{(s-u)_+^{k-1}}{(k-1)!} \frac{(s'-u)_+^{k-1}}{(k-1)!} du \tag{3.3}$$

and

$$K(t,s) = \frac{(t-s)_+^{\ell-1}}{(\ell-1)!} \tag{3.4}$$

with k, ℓ positive integers with $k+\ell = m$. Then $\mathcal{H}_R$ is the space of all functions z on $[0,1]$ with $z^{(\nu)}(0) = 0$, $\nu=0,1,2,\ldots k-1$, $z^{(k)} \in L_2[0,1]$, with

$$\langle z_1, z_2 \rangle_R = \int_0^1 z_1^{(k)}(s)\; z_2^{(k)}(s)\, ds \tag{3.5}$$

It follows that

$$\eta(t,s) = \int_0^1 \frac{(t-u)_+^{m-1}}{(m-1)!} \; \frac{(s-u)_+^{k-1}}{(k-1)!}\; du \tag{3.6}$$

$$Q(t,t') = \int_0^1 \frac{(t-u)_+^{m-1}}{(m-1)!} \; \frac{(t'-u)_+^{m-1}}{(m-1)!}\; du \quad . \tag{3.7}$$

$\mathcal{H}_Q$ is thus the space of all functions u on $[0,1]$ with $u^{(\nu)}(0) = 0$, $\nu = 0,1,2,\ldots m-1$, $u^{(m)} \in L_2[0,1]$, with

$$\langle u_1, u_2 \rangle_Q = \int_0^1 u_1^{(m)}(s)\; u_2^{(m)}(s)\; ds$$

$\eta^*(s,t)$ defined by

$$\eta^*(s,t) = \eta(t,s) \tag{3.8}$$

is the Green's function for the operator L defined by

$$Lz = z^{(k+m)} \tag{3.9}$$

with boundary conditions

$$z^{(\nu)}(0) = 0, \quad \nu = 0,1,2,\ldots k-1 \tag{3.9b}$$

$$z^{(\nu)}(1) = 0 \quad \nu = k, k+1, \ldots k+m-1$$

Thus, z has a representation

$$z(s) = \int_0^1 \eta(t,s)\, \rho(t) dt \tag{3.10}$$

for ρ continuous if and only if $z^{(\nu)}(0) = 0$, $\nu = 0,1,2,\ldots k-1$, $z^{(\nu)}(1) = 0$, $\nu = k, k+1, \ldots k+m-1$, and $z^{(m+k)} = \rho$ continuous.

$P_{V_n} z$, given by

$$P_{V_n} z = (\eta_{t_1}, \eta_{t_2}, \ldots \eta_{t_n}) \; \bar{Q}^{-1}(u(t_1), u(t_2), \ldots u(t_n))' \qquad (1.8)$$

is the solution to the problem:

Find $z \in \mathcal{H}_R$ to minimize

$$\int_0^1 (z^{(k)}(s))^2 \, ds \qquad (3.12)$$

subject to

$$u(t_i) = \int_0^1 \frac{(t_i - s)_+^{\ell-1}}{(\ell-1)!} z(s)\, ds = \int_0^{t_i} d\xi_{\ell-1} \int_0^{\xi_{\ell-1}} d\xi_{\ell-2} \cdots \int_0^{\xi_1} z(s) ds \quad , i=1,2,\ldots n \qquad (3.13)$$

The solution to the problem: Find $\underset{\sim}{u} \in \mathcal{H}_Q$ to minimize

$$\int_0^1 (\underset{\sim}{u}^{(m)}(s))^2 \, ds \qquad (3.14)$$

subject to

$$\underset{\sim}{u}(t_i) = u(t_i) \quad , \qquad i = 1, 2, \ldots n, \qquad (3.15)$$

is $\underset{\sim}{u} = P_{T_n} u$, given by

$$\underset{\sim}{u} = (Q_{t_1}, Q_{t_2}, \ldots Q_{t_n}) \bar{Q}^{-1}(u(t_1), u(t_2), \ldots u(t_n))' \qquad (3.16)$$

that is,

$$\underset{\sim}{u}(t) = (Q(t_1, t), Q(t_2, t), \ldots Q(t_n, t)) \bar{Q}^{-1}(u(t_1), u(t_2), \ldots u(t_n))' . \qquad (3.17)$$

Since

$$\frac{\partial^\ell}{\partial s^\ell} Q(t_i, s) = \eta(t_i, s) \qquad (3.18)$$

we have

$$u^{(\ell)}(s) = (P_{V_n} z)(s) . \tag{3.19}$$

$K^{-1} u$, $u \in \mathcal{H}_Q$ is given by

$$K^{-1} u = u^{(\ell)} \tag{3.20}$$

Thus, (3.19) is an example of the relation

$$K^{-1}(P_{T_n} u) = P_{V_n} z \tag{3.21}$$

whenever

$$Kz = u \tag{3.22}$$

Hence our method of approximate solution of

$$Kz = u \tag{3.23}$$

is equivalent to interpolating u smoothly at the points $u(t_i)$, $i=1,2,\ldots n$ and calculating K^{-1} of the resulting interpolating function.

In general, we have

$$|z(s) - (P_{V_n} z)(s)| \leq \|z - P_{V_n} z\|_R \|R_s - P_{V_n} R_s\|_R \tag{3.24}$$

where, by Theorem (1)

$$\|z - P_{V_n} z\|_R^2 = \|u - P_{T_n} u\|_Q^2$$

$$\leq \sup_t \rho^2(t)(2m+1)\left[\sup_{\xi \neq t} \left|\frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, t)\right|\right.$$

$$\left. + 2 \sup_k \sup_{\xi, t \in I_k} \left|\frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, t)\right|\right] \sum_{k=0}^{N-1} (t_{(k+1)(2m-1)+1} - t_{k(2m-1)+1})^{2m+1}$$

(3.25)

In this case

$$\rho = z^{(m+k)} = u^{(2m)} \tag{3.24}$$

$$\sup_{\xi \neq t} \left| \frac{\partial^{2m}}{\partial \xi^{2m}} Q(\xi, t) \right| = 0 \tag{3.25}$$

$$\sup_{\xi, t \in I_k} \left| \frac{\partial^{2m-1}}{\partial \xi^{2m-1}} Q(\xi, t) \right| = 1 \tag{3.26}$$

and, since $\sum_{k=0}^{N-1} (t_{(k+1)(2m-1)+1} - t_{k(2m-1)+1})^{2m+1} \leq (2m-1)^{2m} \|\Delta\|^{2m}$

we have

$$\|z - P_{V_n} z\|_R \leq 2(2m+1) \sup_s | z^{(m+k)}(s) | (2m-1)^m \|\Delta\|^m \tag{3.27}$$

In Lemma A.1 of the Appendix, it is proved,* for this case, that

$$\| R_s - P_{V_n} R_s \|_R \leq C_{k,m} (m-1)^{k-\frac{1}{2}} \|\Delta\|^{k-\frac{1}{2}} \tag{3.28}$$

where $C_{k,m}$ is given by (A.13). Hence

$$|z(s) - (P_{V_n} z)(s)| \leq 2(2m+1) \sup_s |z^{(m+k)}(s)| \tag{3.29}$$

$$\cdot\, C_{k,m} (2m-1)^m (m-1)^{k-\frac{1}{2}} \|\Delta\|^{m+k-\frac{1}{2}}$$

It appears that Lemma A.1 has an easy generalization to R and K which are appropriate Green's functions, but we do not as yet have the most general conditions on R and K for which error bounds of the form $O(\|\Delta\|)^{m+k-\frac{1}{2}}$ obtain.

*The method of proof assumes a bounded mesh ratio.

Now

$$|z(s) - (P_{V_n} z)(s)| = |u^{(\ell)}(s) - \underset{\sim}{u}^{(\ell)}(s)| \qquad , \ \ell = 0,1,2,\ldots m-1.$$

$$\leq 2(2m+1) \sup_s |u^{(2m)}(s)| \, C_{k,m} (m-1)^{k-\frac{1}{2}} \|\Delta\|^{2m-\ell-\frac{1}{2}} \qquad (3.30)$$

Since $\underset{\sim}{u}(s)$ is the solution of the minimization problem of (3.14) and (3.15), it is well known that $\underset{\sim}{u}(s)$ is the (unique) 2m-1 st degree polynomial spline of interpolation to $u(t_i)$, $t = 1,2,\ldots n$, satisfying the boundary conditions

$$\underset{\sim}{u}^{(\nu)}(0) = 0, \quad \nu = 0,1,2,\ldots m-1$$

We have thus proved in (3.30) the following Theorem, which is typical of convergence theorems to be found in the spline literature. (see e.g. [1])

<u>Theorem 5.</u>

Let $u(t) \in C^{2m}[0,1]$, with $u^{(\nu)}(0) = 0$, $\nu = 0,1,2,\ldots m-1$, $u^{(\nu)}(1) = 0$, $\nu = m, m+1, \ldots 2m-1$, and let $\underset{\sim}{u}(t)$ be the 2m-1st degree polynomial spline of interpolation to $u(t)$ at $t = t_1, t_2, \ldots t_n$ satisfying the boundary conditions $\underset{\sim}{u}^{(\nu)}(0) = 0$, $\nu = 0,1,2,\ldots m-1$. Then

$$|u^{(\ell)}(s) - \underset{\sim}{u}^{(\ell)}(s)| \leq 2(2m+1) \sup_s | u^{(2m)}(s) \; C_{k,m} (m-1)^{k-\frac{1}{2}} \|\Delta\|^{2m-\ell-\frac{1}{2}} \qquad (3.31)$$

$$\ell = 1,2,\ldots m-1.$$

<u>Appendix.</u>

This appendix is devoted to the proof of Lemma A.1 used in the Example of Section 3. The Lemma is not necessarily new, but is typical of convergence theorems in the spline literature, see e.g. [1].

<u>Lemma A.1</u>

Let

$$R(s,s') = \int_0^1 \frac{(s-u)_+^{k-1}}{(k-1)!} \quad \frac{(s'-u)_+^{k-1}}{(k-1)!} \, du$$

and $K(t,s) = \dfrac{(t-s)_+^{\ell-1}}{(\ell-1)!}$, with $k + \ell = m$. Let

$$\sup_i |t_{i+1} - t_i| \, / \, \inf_i |t_{i+1} - t_i| = a < \infty \tag{A.1}$$

Then

$$\| R_s - P_{V_n} R_s \|_R \leq C_{k,m} \; m^{k-\frac{1}{2}} \, \| \Delta \|^{k-\frac{1}{2}} \tag{A.2}$$

where

$$C_{k,m} = \left[\frac{1}{(k-1)!} + \sum_{\tau=1}^{m-1} \frac{1}{(m-1)!} \quad \frac{(\tau a)^{m-1}}{\prod\limits_{\substack{\ell \neq \tau \\ \ell=0,1,2,\ldots m-1}} |\tau - \ell|} \binom{m-1}{\ell} \right]$$

<u>Proof.</u>

$\eta_t(s)$ is given by

$$\eta_t(s) = \int_0^1 \frac{(t-u)_+^{m-1}}{(m-1)!} \quad \frac{(s-u)_+^{k-1}}{(k-1)!} \, du \tag{A.3}$$

For the remainder of the proof we consider s fixed. Then for any $\{d_i(s)\}_{i=1}^n$,

$$\| R_s - P_{V_n} R_s \|_R^2 \leq \| R_s - \sum_{i=1}^n d_i(s)\, \eta_{t_i} \|_R^2 \tag{A.4}$$

$$= \int_0^1 \frac{(s-u)_+^{k-1}}{(k-1)!} \frac{(s-u)_+^{k-1}}{(k-1)!} du - 2 \sum_{i=1}^n d_i(s) \int_0^1 \frac{(t_i-u)_+^{m-1}}{(m-1)!} \frac{(s-u)_+^{k-1}}{(k-1)!}$$

$$+ \sum_{i=1}^n \sum_{j=1}^n d_i(s)\, d_j(s) \int_0^1 \frac{(t_i-u)_+^{m-1}}{(m-1)!} \frac{(t_j-u)_+^{m-1}}{(m-1)!} du$$

$$= \int_0^1 \left(\frac{(s-u)_+^{k-1}}{(k-1)!} - \sum_{i=1}^n d_i(s) \frac{(t_i-u)_+^{m-1}}{(m-1)!} \right)^2 du \tag{A.5}$$

Suppose $t_j \leq s \leq t_{j+1}$. Set $d_i(s) = 0$, except for $i = j, j+1, \ldots$ $j+m-1$. Then (A.5) can be written

$$\int_0^{t_j} \left(\frac{(s-u)_+^{k-1}}{(k-1)!} - \sum_{i=j}^{j+m-1} d_i(s) \frac{(t_i-u)^{m-1}}{(m-1)!} \right)^2 du + \int_{t_j}^{t_{j+m-1}} \left(\frac{(s-u)_+^{k-1}}{(k-1)!} - \sum_{i=j}^{j+m-1} d_i(s) \frac{(t_i-u)_+^{m-1}}{(m-1)!} \right)^2 du \tag{A.6}$$

The first term may be made to vanish by choosing the $\{d_i(s)\}_{i=j}^{j+m-1}$ so that the coefficient of u^ℓ is 0, for $\ell = 0, 1, 2, \ldots m-1$.

This is accomplished by letting $\{d_i(s)\}_{i=j}^{j+m-1}$ satisfy

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_j & t_{j+1} & \cdots & t_{j+m-1} \\ \vdots & \vdots & & \vdots \\ \\ \frac{t_j^{m-1}}{(m-1)!} & \frac{t_{j+1}^{m-1}}{(m-1)!} & \cdots & \frac{t_{j+m-1}^{m-1}}{(m-1)!} \end{pmatrix} \begin{pmatrix} d_j(s) \\ d_{j+1}(s) \\ \vdots \\ \\ d_{j+m-1}(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ s \\ \frac{s^2}{2!} \\ \vdots \\ \frac{s^{k-1}}{(k-1)!} \end{pmatrix} \qquad \text{(A. 7)}$$

We may get an explicit expression for $d_{j+\tau}(s)$, $\tau = 0, 1, 2, \ldots m-1$, by noting that $c_{j+\tau}(s)$, $\tau = 0, 1, 2, \ldots m-1$ defined by

$$\begin{pmatrix} c_j(s) \\ c_{j+1}(s) \\ \vdots \\ \\ c_{j+m-1}(s) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_j & t_{j+1} & \cdots & t_{j+m-1} \\ \vdots & \vdots & & \\ \\ \frac{t_j^{m-1}}{(m-1)!} & \frac{t_{j+1}^{m-1}}{(m-1)!} & \cdots & \frac{t_{j+m-1}^{m-1}}{(m-1)!} \end{pmatrix} \begin{pmatrix} 1 \\ s \\ \frac{s^2}{2!} \\ \vdots \\ \frac{s^{m-1}}{(m-1)!} \end{pmatrix} \qquad \text{(A. 8)}$$

is the (m-1)st degree polynomial which is 1 at $s = t_{j+\tau}$ and is 0 at $s = t_{j+\ell}$, $\ell = 0, 1, 2, \ldots m-1$, $\ell \neq \tau$. Thus

$$c_{j+\tau}(s) = \frac{\prod_{\ell \neq t} (s - t_{j+\ell})}{\prod_{\ell \neq \tau} (t_{j+\tau} - t_{j+\ell})} \quad , \quad \tau = 0, 1, 2, \ldots m-1 \qquad \text{(A. 9)}$$

and

$$d_{j+\tau}(s) = \frac{d^\ell}{ds^\ell} \quad c_{j+\tau}(s) = \frac{1}{\underset{\substack{\ell \neq \tau \\ \ell=0,1,2\ldots m-1}}{\Pi}(t_{j+\tau}-t_{j+\ell})} \sum_{\substack{i_1, i_2, \ldots i_\ell \\ i_\xi \neq \tau}} \prod_{\substack{\nu \neq i_1, i_2, \ldots i_\ell \\ \nu \neq \tau}} (s-t_{j+\nu}) \tag{A.10}$$

where the sum is over all $\binom{m-1}{\ell}$ ways the indices $\{0,1,2,\ldots \tau-1, \tau+1, \ldots m-1\}$ may be selected ℓ at a time. With this choice of $\{d_{j+\tau}(s)\}_{\tau=0}^{m-1}$

(A.6) becomes

$$\int_{t_j}^{t_{j+m-1}} \left[\frac{(s-u)_+^{k-1}}{(k-1)!} - \sum_{\tau=1}^{m-1} \frac{(t_{j+\tau}-u)_+^{m-1}}{(m-1)!} \frac{1}{\underset{\substack{\ell \neq \tau \\ \ell=0,1,2,\ldots m-1}}{\Pi}(t_{j+\tau}-t_{j+\ell})} \sum_{\substack{i_1, i_2, \ldots i_\ell \\ i_\xi \neq \tau}} \prod_{\substack{\nu \neq i_1, i_2, \ldots i_\ell \\ \nu \neq \tau}} (s-t_{j+\nu}) \right]^2 du \tag{A.11}$$

Since $|(t_{j+\tau}-u)_+ / (t_{j+\tau}-t_{j+\ell})| \leq \frac{\tau}{|\tau-\ell|} a$ and $t_j \leq s \leq t_{j+1}$,

we have that (A.11) is certainly bounded by

$$C_{k,m}^2 (m-1)^{2k-1} \|\Delta\|^{2k-1}. \tag{A.12}$$

where

$$C_{k,m} = \left[\frac{1}{(k-1)!} + \sum_{\tau=1}^{m-1} \frac{1}{(m-1)!} \frac{(\tau a)^{m-1}}{\underset{\ell \neq \tau}{\Pi}|\tau-\ell|} \binom{m-1}{\ell} \right] \tag{A.13}$$

and

$$\|R_s - P_{V_n} R_s\|_R = O(\|\Delta\|^{k-\frac{1}{2}}) \tag{A.14}$$

Remark. The case of general R may be converted to a problem in the rate of convergence of an approximation in the L_2 norm as follows. Assume R has a factorization of the form

$$R(s,s') = \int_U G(s,u)\, G(s',u)\, du \tag{A.15}$$

where U is, say, a given interval, and assume that G and K are sufficiently regular to allow interchanges of the order of integration below. Observe that

$$\langle R_s, R_s \rangle_R = R(s,s) = \int_U G^2(s,u)\, du$$

$$\langle R_s, \eta_{t_i} \rangle_R = \eta_{t_i}(s) = \int_S K(t_i, s')\, R(s,s')\, ds'$$

$$= \int_U du\, G(s,u) \int_S K(t_i, v)\, G(v,u)\, dv \tag{A.17}$$

and

$$\langle \eta_{t_i}, \eta_{t_j} \rangle_R = \int_S \int_S K(t_i, s)\, R(s,s')\, K(t_j, s')\, ds\, ds'$$

$$= \int_U du \int_S K(t_i, v) G(v,u) dv \int_S K(t_j, v') G(v',u) dv' \tag{A.18}$$

Then, for any $\{d_i(s)\}_{i=1}^n$,

$$\| R_s - P_{V_n} R_s \|_R^2 \leq \| R_s - \sum_{i=1}^n d_i(s)\, \eta_{t_i} \|_R^2$$

$$= \int_U [G(s,u) - \sum_{i=1}^n d_i(s) \int_S K(t_i, v)\, G(v,u) dv]^2\, du \tag{A.16}$$

The problem of finding error bounds is then one of finding $d_i(s)$ so that $\{\int_S K(t_i, v)\, G(v, \cdot)\, dv\}_{i=1}^n$ well approximate $G(s, \cdot)$ in the L_2 norm.

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