Department of Statistics

The University of Wisconsin
Madison, Wisconsin

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A NOTE ON INTERPOLATION OVER ALL THE INTEGERS

by

Grace Wahba

Typist: Mary Ann Clark

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Abstract

Let L_m be the differential operator defined by $L_m f = \prod_{\nu=1}^{m} (D + \alpha_\nu) f$, where $\{\alpha_\nu\}_{\nu=1}^{m}$ are m distinct positive real numbers and D is differentiation. Let $\mathcal{H}_R = \{f : f^{(r)}$ is absolutely continuous on $(-\infty, \infty)$, $r = 0, 1, 2, \ldots m-1$, $f \in L_2(-\infty, \infty)$, $L_m f \in L_2(-\infty, \infty)\}$. We solve the constrained minimization problem: Find $h \in \mathcal{H}_R$ to minimize

$$\int_{-\infty}^{\infty} [(L_m h)(s)]^2 \, ds$$

subject to the constraints $h(j) = f_j$, $j = \cdots -1, 0, 1, \cdots$, where $\{f_j\}_{j=-\infty}^{\infty}$ is a given sequence satisfying

$$\sum_{j=-\infty}^{\infty} f_j^2 < \infty$$

It is shown that $\sum_{j=-\infty}^{\infty} f_j^2 < \infty$ is a necessary and sufficient condition for the existence of a (necessarily unique) solution. If f is any element in $\mathcal{H}_R$ satisfying $f(j) = f_j$, $j = \cdots -1, 0, 1, \cdots$, pointwise error bounds on

$$|f(s) - h(s)|$$

where h is the solution to the constrained minimization problem, are given. A formula for h is easily given by applying standard techniques related to the interpolation of a zero mean stationary Gaussian stochastic process with spectral density $|P(\lambda)|^{-2}$, $P(\lambda) = \prod_{\nu=1}^{m} (i\lambda + \alpha_\nu)$. Some generalizations are discussed.

A Note on Interpolation Over All the Integers

1. Introduction

Recently, there has been some interest in interpolation, when the data to be interpolated is given at more than a finite number of points [2][8], although this interest goes back many years, (See, for example [7]). The problems of existence and uniqueness, not to mention construction, of interpolating functions satisfying some optimality criterion may be quite difficult. In this note we consider what may be viewed as one of the simplest special cases of this type of problem. The main reason for considering this case, is that it is not difficult to explicitly construct and study the (unique) solution, by elementary techniques. It seems however, that this case has been overlooked in the approximation theory literature.

Let L_m be the differential operator defined by $L_m f = \prod_{\nu=1}^{m} (D + \alpha_\nu) f$, where $\{\alpha_\nu\}_{\nu=1}^{m}$ are m distinct postive real numbers and D is differentiation. Let $\mathcal{H}_R = \{f : f^{(r)}$ is absolutely continuous on $(-\infty, \infty)$, $r = 0, 1, 2, \ldots m-1$, $f \in L_2(-\infty, \infty)$, $L_m f \in L_2(-\infty, -)\}$. We solve the constrained minimization problem: Find $h \in \mathcal{H}_R$ to minimize

$$\int_{-\infty}^{\infty} [(L_m h)(s)]^2 \, ds \tag{1}$$

subject to the constraints $h(j) = f_j$, $j = \ldots -1, 0, 1, \ldots$, where $\{f_j\}_{j=-\infty}^{\infty}$ is a given sequence satisfying

$$\sum_{j=-\infty}^{\infty} f_j^2 < \infty \tag{2}$$

It is shown that $\sum_{j=-\infty}^{\infty} f_j^2 < \infty$ is a necessary and sufficient condition for the existence of a (necessarily unique) solution. If f is any element in $\mathcal{H}_R$ satisfying $f(j) = f_j$, $j = \ldots -1, 0, 1, \ldots,$ pointwise error bounds on

$$|f(s) - h(s)|$$

where h is the solution to the constrained minimization problem, are given. A formula for h is easily given by applying standard techniques related to the interpolation of a zero mean stationary Gaussian stochastic process with spectral density $|P(\lambda)|^{-2}$, $P(\lambda) = \prod_{\nu=1}^{m} (i\lambda + \alpha_\nu)$. Some generalizations are discussed, in particular where L_m is replaced by a differential operator of infinite order.

2. Construction of the Solution

Define $G(s,u)$ by

$$G(s,u) = \sum_{\nu=1}^{m} \frac{e^{-\alpha_\nu (s-u)}}{\prod_{j \neq \nu} (\alpha_j - \alpha_\nu)} \qquad s > u \tag{3}$$

$$= 0 \qquad s < u$$

For $p \in L_2(-\infty, \infty)$ define the operator G by

$$(G\,p)(s) = \int_{-\infty}^{\infty} G(s,u)\, p(u)\, du \tag{4}$$

Letting

$$f\,(s) = (G\,p)\,(s) \tag{5}$$

it is straightforward to check that

$$L_m f = p \qquad \text{a.e.} \tag{6}$$

We also have that $f^{(r)}$ is absolutely continuous, for $r = 0, 1, 2, \ldots m-1$.

We shall seek a solution to the constrained minimization problem in the class of functions given by the range of G, with domain $L_2(-\infty, \infty)$.

Another way of writing (5) is

$$f = g * p$$

where

$$g(\tau) = \sum_{\nu=1}^{m} \frac{e^{-\alpha_\nu \tau}}{\prod_{j \neq \nu} (\alpha_j - \alpha_\nu)} \qquad \tau > 0 \tag{7}$$

$$= 0 \qquad \text{otherwise}$$

Since the Fourier transform ϕ_g of g is given by

$$\phi_g(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda\tau} g(\tau)d\tau = \frac{1}{P(\lambda)} \tag{8}$$

where

$$P(\lambda) = \prod_{\nu=1}^{m} (i\lambda + \alpha_\nu) \tag{9}$$

we have that f is in the range of G if and only if it has a Fourier transform which is of the form

$$\phi_f(\lambda) = \frac{\phi_p(\lambda)}{\prod_{\nu=1}^{m} (i\lambda + \alpha_\nu)} \tag{10}$$

where $\phi_p \in L_2(-\infty, \infty)$.

Since $p \neq 0$ as an element of $L_2(-\infty, \infty)$ implies that $f \neq 0$, we may make a Hilbert space out of the range of G, with inner product $\langle \cdot , \cdot \rangle$ defined by

$$\langle f_1, f_2 \rangle = \int_{-\infty}^{\infty} p_1(u)p_2(u)du = \int_{-\infty}^{\infty} (L_m f_1(t))(L_m f_2(t))\, dt \tag{11}$$

where

$$f_i = g * p_i \,, \qquad i=1,2 \tag{12}$$

It is easy to check that this is a Hilbert space, which we shall call $\mathcal{H}_R$. $\mathcal{H}_R$ is a reproducing kernel Hilbert space with reproducing kernel

$$R(t, t') = \int_{-\infty}^{\infty} G(t, u)\, G(t', u)\, du \tag{13}$$

That is, to say, the function $R_t(\cdot)$ defined by

$$R_t(t') = R(t,t') \tag{14}$$

is in $\mathcal{H}_R$, for $\forall\, t$, $t \in (-\infty, \infty)$, and

$$\langle f, R_t \rangle = f(t) \;, \quad f \in \mathcal{H}_R, \quad t \in (-\infty, \infty) \tag{15}$$

The reader is referred to [1],[3], and [6] for examples of the use of reproducing kernel spaces in approximation theory.

We may also characterize $\mathcal{H}_R$ as

$$\mathcal{H}_R = \{f\colon f^{(r)} \text{ absolutely continuous, } r=0,1,2,\dots m-1, \tag{16}$$

$$f \in L_2(-\infty,\infty), L_m f \in L_2(-\infty,\infty)\}$$

We now reformulate the problem as follows:

Find $h \in \mathcal{H}_R$ to minimize $\|h\|^2$ subject to

$$\langle h, R_j \rangle = f_j \;, \quad j = \cdots -1, 0, 1, \cdots \tag{17}$$

Now, let $\overline{\mathcal{H}}_R$ be the subspace spanned by the elements $\{R_j\}_{j=-\infty}^{\infty}$. It then follows that there exists a solution if and only if there exists an element $f \in \mathcal{H}_R$ satisfying

$$\langle f, R_j \rangle = f_j, \quad i = \cdots -1, 0, 1, \cdots \quad . \tag{18}$$

For $f \in \mathcal{H}_R$, let $\hat{f}$ be the projection of f onto $\overline{\mathcal{H}}_R$. Then, if an f satisfying (18) exists, the solution h to the minimization problem is unique and is given by $\hat{f}$. We will defer until later conditions on $\{f_i\}_{i=-\infty}^{\infty}$ such that an f satisfying (18) exists. We now assume that it does, and proceed to derive a formula for $\hat{f}(s_*) = h(s_*)$, $s_* \in (-\infty, \infty)$.

Fix $s_* \in (-\infty, \infty)$ and let

$$\hat{R}_{s_*} = \sum_{j=-\infty}^{\infty} b_j(s_*) R_j \tag{19}$$

where the $\{b_j(s_*)\}$ are to be found. Once the $\{b_j(s_*)\}_{j=-\infty}^{\infty}$ are known, a formula for $\hat{f}(s_*)$ is given as follows:

$$\begin{aligned} \hat{f}(s_*) = \left\langle \hat{f}, R_{s_*} \right\rangle &= \left\langle f, \hat{R}_{s_*} \right\rangle \\ &= \sum_{j=-\infty}^{\infty} b_j(s_*) \left\langle R_j, f \right\rangle \\ &= \sum_{j=-\infty}^{\infty} b_j(s_*) f_j \end{aligned} \tag{20}$$

The family $\{R_j\}_{j=-\infty}^{\infty}$ is linearly independent, this is a consequence of a well known theorem in time series, whose discussion we will also defer until later. The $\{b_j(s_*)\}_{j=-\infty}^{\infty}$ are, as a consequence, uniquely determined, for each s_*, and they minimize

$$\| R_{s_*} - \sum_{j=-\infty}^{\infty} b_j(s_*) R_j \|^2 \tag{21}$$

Expression (21) may be written, with the aid of (14) and (15) as

$$R(s_*, s_*) - 2 \sum_{j=-\infty}^{\infty} b_j(s_*) R(j, s_*) + \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} b_j(s_*) b_k(s_*) R(j,k) \tag{22}$$

The normal equations, which must be satisfied for (22) to be a minimum, are then

$$\sum_{k=-\infty}^{\infty} R(j,k) b_k(s_*) = R(j,s_*) , \qquad j = \cdots -1, 0, 1, \cdots \tag{23}$$

From (13) we may calculate that

$$R(t,t') = \sum_{\nu=1}^{m} \theta_\nu e^{-\alpha_\nu |t-t'|} \tag{24}$$

where

$$\theta_\nu = \sum_{\mu=1}^{m} \frac{c_\mu c_\nu}{(\alpha_\mu + \alpha_\nu)} , \qquad c_\mu = \Big(\prod_{j \neq \mu} (\alpha_j - \alpha_\mu) \Big)^{-1} , \ \mu = 1, 2 \ldots m$$

and hence

$$R(j,k) = r(j-k) \tag{25a}$$

where

$$r(\tau) = \sum_{\nu=1}^{m} \theta_\nu e^{-\alpha_\nu |\tau|} \qquad \tau \in (-\infty, \infty) \tag{25b}$$

Hence (22) becomes

$$\sum_{k=-\infty}^{\infty} r(j-k) \, b_k(s_*) = r(j-s_*) \qquad j = \ldots -1, 0, 1, \cdots \tag{26}$$

It is clear from (26) that $b_k(s_*) = b_{k+n}(s_* + n)$ for n an integer. Hence, it is only necessary to solve (26) for $0 < s_* < 1$.

Equation (26) is a standard convolution equation which is solved as follows.

Let

$$\psi(\lambda) = \sum_{j=-\infty}^{\infty} e^{-ij\lambda} \, r(j)$$

$$= \sum_{j=-\infty}^{\infty} e^{-ij\lambda} \sum_{\nu=1}^{m} \theta_\nu e^{-\alpha_\nu |j|} \tag{27}$$

$$= \sum_{\nu=1}^{m} \theta_\nu \frac{(1-a_\nu^2)}{(1-a_\nu e^{-i\lambda})(1-a_\nu e^{i\lambda})} \qquad a_\nu = e^{-\alpha_\nu}, \ \nu = 1, 2, \ldots, m$$

$$\psi_{s_*}(\lambda) = \sum_{j=-\infty}^{\infty} e^{-ij\lambda}\, r(j-s_*)$$

$$= \sum_{j=-\infty}^{\infty} e^{-ij\lambda} \sum_{\nu=1}^{m} \theta_\nu\, e^{-\alpha_\nu |j-s_*|} \tag{28}$$

$$= \sum_{\nu=1}^{m} \theta_\nu \frac{e^{-\alpha_\nu s_*}(1-a_\nu e^{-i\lambda}) + e^{+\alpha_\nu s_*}[(1-a_\nu e^{i\lambda})-(1-a_\nu e^{i\lambda})(1-a_\nu e^{-i\lambda})]}{(1-a_\nu e^{-i\lambda})(1-a_\nu e^{i\lambda})}$$

and

$$b(\lambda) = \sum_{j=-\infty}^{\infty} e^{-ij\lambda}\, b_j(s_*) \tag{29}$$

By multiplying the left and right hand sides of the jth equation in (23) by $e^{-ij\lambda}$ and summing over j, we obtain

$$\psi(\lambda)\, b(\lambda) = \psi_{s_*}(\lambda) \tag{30}$$

Since $\psi(\lambda)$ has no real zeroes, we may write

$$b(\lambda) = \psi_{s_*}(\lambda) / \psi(\lambda) \tag{31}$$

and

$$b_j(s_*) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij\lambda} \left(\psi_{s_*}(\lambda)/\psi(\lambda)\right) d\lambda \tag{32}$$

Some algebraic manipulation on (27), (28) and (32) results in

$$b_j(s_*) = \frac{1}{2\pi} \sum_{\nu=1}^{m} \theta_\nu e^{-\alpha_\nu s_*} \int_{-\pi}^{\pi} \left[\frac{(1-a_\nu e^{-i\lambda}) \prod_{j \neq \nu} (1-a_j e^{i\lambda})(1-a_j e^{-i\lambda})}{\sum_{\mu=1}^{m} \theta_\mu (1-a_\mu^2) \prod_{k \neq \mu} (1-a_k e^{i\lambda})(1-a_k e^{-i\lambda})} \right] d\lambda$$

$$+ \frac{1}{2\pi} \sum_{\nu=1}^{m} \theta_\nu e^{\alpha_\nu s_*} \int_{-\pi}^{\pi} \left[\frac{a_\nu e^{-i\lambda}(1-a_\nu e^{i\lambda}) \prod_{j \neq \nu} (1-a_j e^{i\lambda})(1-a_j e^{-i\lambda})}{\sum_{\mu=1}^{m} \theta_\mu (1-a_\mu^2) \prod_{k \neq \mu} (1-a_k e^{i\lambda})(1-a_k e^{-i\lambda})} \right] d\lambda \qquad (33)$$

$$j = \ldots -1, 0, 1, \ldots$$

$$0 < s_* < 1$$

$$b_j(s_*) = b_{j-m}(s_*-m) \qquad m < s_* < m+1$$

3. Existence

We now address ourselves to the question of conditions on $\{f_i\}_{i=-\infty}^{\infty}$ so that there exists an $f \in \mathcal{H}_R$ satisfying

$$\langle f, R_j \rangle = f_j \,, \qquad j = \ldots -1, 0, 1, \ldots \qquad (34)$$

The necessary and sufficient conditions are, that $\hat{f}$ defined by

$$\hat{f}(s_*) = \sum_{j=-\infty}^{\infty} b_j(s_*) f_j \tag{35}$$

with $b_j(s_*)$ given by (33) is in $\mathcal{H}_R$, since this is in fact the projection onto $\bar{\mathcal{H}}_R$ of any element f in $\mathcal{H}_R$ satisfying (18). Now, by (32), we may write

$$b_j(s_*) = \sum_{k=-\infty}^{\infty} r^{(j-k)} R(k, s_*) \tag{36}$$

where $r^{(\tau)}$ is given by

$$r^{(\tau)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\lambda\tau} \frac{d\lambda}{\psi(\lambda)} . \tag{37}$$

It may be verified that

$$L_m R_k(s) = L_m R(k, s) = G(k, s) \tag{38}$$

Thus

$$L_m b_j(s) = \sum_{k=-\infty}^{\infty} r^{(j-k)} G(k, s) \tag{39}$$

and

$$\|\hat{f}\|^2 = \int_{-\infty}^{\infty} [(L_m \hat{f})(s)]^2 ds$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f_j f_k \int_{-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} r^{(j-\ell)} G(\ell, s) G(n, s) r^{(k-n)} ds \tag{40}$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f_j f_k \sum_{\ell=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} r^{(j-\ell)} R(\ell, n) r^{(k-n)}$$

$$= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f_j f_k \sum_{\ell=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} r^{(j-\ell)} r(\ell-n) r^{(k-n)}$$

Letting

$$F(\lambda) = \sum_{j=-\infty}^{\infty} e^{-i\lambda j} f_j, \tag{41}$$

(40) may be written

$$\int_{-\infty}^{\infty} [(L_m \hat{f})(s)]^2 \, ds = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|F(\lambda)|^2}{\psi(\lambda)} \, d\lambda \tag{42}$$

Since $\psi(\lambda)$ is bounded above and below between positive limits, the finiteness of (42) is equivalent to

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(\lambda)|^2 \, d\lambda = \sum_{j=-\infty}^{\infty} f_j^2 < \infty \; . \tag{43}$$

4. Pointwise Error Bounds

We may use (42) to put a pointwise error bound on interpolation of this kind. Let $f \in \mathcal{H}_R$ with

$$\langle f, R_j \rangle = f_j \, , \qquad j = \ldots -1, 0, 1, \ldots$$

and

$$F(\lambda) = \sum_{j=-\infty}^{\infty} e^{-i\lambda j} f_j$$

Then

$$|f(s_*) - \hat{f}(s_*)|^2 = | \langle f - \hat{f}, R_{s_*} \rangle |^2 = | \langle f - \hat{f}, R_{s_*} - \hat{R}_{s_*} \rangle |^2$$

$$\leq \| f - \hat{f} \|^2 \, \| R_{s_*} - \hat{R}_{s_*} \|^2$$

$$= \left[\int_{-\infty}^{\infty} [(L_m f)(s)]^2 \, ds - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|F(\lambda)|^2}{\psi(\lambda)} \, d\lambda \right] \times$$

$$\left[R(s_*, s_*) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\psi_{s_*}(\lambda)|^2}{\psi(\lambda)} \, d\lambda \right] \quad (44)$$

$$= \left[\int_{-\infty}^{\infty} |P(\lambda)|^2 |\phi_f(\lambda)|^2 d\lambda - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|F(\lambda)|^2}{\psi(\lambda)} \, d\lambda \right] \times$$

$$\left[\int_{-\infty}^{\infty} \left| \frac{1}{|P(\lambda)|^2} \right. d\lambda - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\psi_{s_*}(\lambda)|^2}{\psi(\lambda)} \, d\lambda \right]$$

5. Linear Independence of the $\{R_j\}_{j=-\infty}^{\infty}$

The reader familiar with the literature of Time Series Analysis will recognize a similarity between the interpolation discussed here and least-squares interpolation on stochastic processes. Let $\{X(t), -\infty < t < \infty\}$ be a zero mean Gaussian stochastic processes with

$$E\, X(t)\, X(t') = R(t, t')$$

where $R(t,t')$ is given by (13). $\{X(t), t \in (-\infty, \infty)\}$ is a stationary process with spectral density $|P(\lambda)|^{-2}$, where $P(\lambda)$ is given by (9).

$\hat{X}(s_*)$, defined by

$$\hat{X}(s_*) = E\{X(s^*) \mid X(j),\ j = \ldots -1, 0, 1, \ldots\}$$

is well known to be given by

$$\hat{X}(s_*) = \sum_{j=-\infty}^{\infty} b_j(s_*) X(j)$$

where the $\{b_j(s_*)\}$ minimize (21).

The assertion that the $\{R_j\}_{j=-\infty}^{\infty}$ are linearly independent as elements of $\mathcal{H}_R$ is equivalent to the assertion that $\{X(j)\}_{j=-\infty}^{\infty}$ are linearly independent as random variables. The reader is referred to [6] for a discussion of the relations between $\mathcal{H}_R$ and the Hilbert space spanned by $\{X(t),\ -\infty < t < \infty\}$.

If a stationary process $\{X(t),\ -\infty < t < \infty\}$ has a spectral density function $|P(\lambda)|^{-2}$, (where $|P(\lambda)|^{-2}$ is here any measurable function), then it is well known that the condition

$$\int_{-\infty}^{\infty} \frac{\log |P(\lambda)|^{-2}}{1+\lambda^2}\, d\lambda < \infty \tag{45}$$

is necessary and sufficient that the random variable $X(t_o)$ is linearly independent of all the random variables $\{X(t),\ t \le t_o - \delta,\ t \ge t_o + \delta,\ \delta > 0\}$, any t_o. See, for example [9], p 189. The condition (45) is satisfied by $P(\lambda)$ given by (9), thus, in this case the $\{X(j)\}_{j=-\infty}^{\infty}$ are linearly independent.

6. Generalizations

i) Non-distinct $\{\alpha_\nu\}_{\nu=1}^{m}$

We remark that our problem may be solved in a straight forward, albeit somewhat tedious manner by the method described, if the $\{\alpha_\nu\}_{\nu=1}^m$ are not required to be distinct. However, we do not apply this procedure if some of the $\{\alpha_\nu\}_{\nu=1}^m$ are zero, since we are dividing by $P(\lambda)$, which would then not be strictly positive. We refer the reader to [7] and [8] for a discussion of the case $L_m = D^m$.

ii) Rational Spectral Densities

The constrained minimization problem in a reproducing kernel Hilbert space may also be solved explicitly in the same manner if L_m is replaced by the operator A defined by

$$Af = \prod_{\nu=1}^{m} (D + \alpha_\nu)(Bf) \tag{46a}$$

where

$$(Bf)(s) = \int_{-\infty}^{\infty} B(s,u)\, f(u)\, du \tag{46b}$$

with

$$B(s,u) = \sum_{\nu=1}^{q} \frac{e^{-\beta_\nu (s-u)}}{\prod_{j \neq \nu} (\beta_j - \beta_\nu)} \qquad s \geq u \tag{46c}$$

$$= 0 \qquad s < u$$

where $q < m$ and $\{\beta_\nu\}_{\nu=1}^q$ are distinct positive integers.

In this case the spectral density $|P(\lambda)|^{-2}$ is replaced by $|Q(\lambda)|^2 \, |P(\lambda)|^{-2}$, where

$$Q(\lambda) = \prod_{\nu=1}^{q} (i\lambda + \beta_\nu),$$

and $R(s, s')$ is replaced by

$$K(s,s') = \int_{-\infty}^{\infty} e^{i\lambda(s-s')} \left| \frac{\prod_{\nu=1}^{q} (i\lambda + \beta_\nu)}{\prod_{\nu=1}^{m} (i\lambda + \alpha_\nu)} \right|^2 d\lambda \tag{47}$$

$\mathcal{H}_R$ is replaced by $\mathcal{H}_K = \{f\colon f^{(r)}$ absolutely continuous, $f \in L_2(-\infty, \infty)$, $r = 0, 1, 2, \ldots m-q$, $Af \in L_2(-\infty, \infty)\}$. An explicit solution resembling (33) can be found. We omit the details.

iii) Polya Frequency Functions

Let now $P(\lambda)$ be of the form

$$P(\lambda) = \prod_{\nu=1}^{\infty} (1 + \delta_\nu i\lambda) \tag{48}$$

where $\delta_\nu > 0$ and $\sum_{\nu=1}^{\infty} \delta_\nu < \infty$. Then

$$\int_{-\infty}^{\infty} \frac{\log |P(\lambda)|^{-2}}{1+\lambda^2} \, d\lambda < \infty \tag{49}$$

Let

$$g(\tau) = \int_{-\infty}^{\infty} \frac{e^{i\lambda\tau}}{P(\lambda)} \, d\lambda \qquad (50)$$

$g(\tau)$ is an example of a Polya frequency function, with $g(\tau) = 0$, $\tau \leq 0$. There is an extensive literature on Polya frequency functions. See Karlin [5], Chap. 7, and references to the works of Schoenberg, listed there.

If $G(s,u)$ is defined by

$$G(s,u) = g(s-u) \qquad (51)$$

and the operator G is defined as in (4), we again make a reproducing kernel Hilbert space $\mathcal{H}_R$ out of the range of G with domain $L_2(-\infty, \infty)$. This space has a reproducing kernel $R(s,s')$ of the form (13) with G given by (51). Let the n^{th} order differential operator L_n be defined by

$$L_n f = \prod_{\nu=1}^{n} (1 + \delta_\nu D) f \qquad (52)$$

If

$$f(s) = (Gp)(s) \qquad (53)$$

then the Hirschman-Widder Theory (see [4], especially Theorem 5.3.b) tells us that

$$\lim_{n \to \infty} (L_n f)(s) = p(s) \qquad (54)$$

if $p \in L_1(-\infty, \infty)$ and s is a point of continuity of p. We are then justified in describing the norm in $\mathcal{H}_R$ as

$$\| f \|^2 = \int_{-\infty}^{\infty} [(Lf)(s)]^2 \, ds \tag{55}$$

where

$$Lf = \prod_{\nu=1}^{\infty} (1 + \delta_\nu D) f = \lim_{n \to \infty} L_n f \quad \text{a.e.} \tag{56}$$

By (54) we can assert the validity of (38) with L_m in (38) replaced by L, and the present R. Thus, if

$$\sum_{j=-\infty}^{\infty} f_i^2 < \infty ,$$

there is a unique solution to the problem :

Find $h \in \mathcal{H}_R$ to minimize

$$\int_{-\infty}^{\infty} [(Lh)(s)]^2 \, ds \tag{57}$$

subject to the constraints $h(j) = f_j$, $j = \ldots -1, 0, 1, \ldots$.

and it is given by (20) upon solving the system (23).

(iv) Further generalities

The most general discussion of this problem that we know of goes as follows. Let $P(\lambda)$ be any (measurable) function satisfying

$$\int_{-\infty}^{\infty} \frac{1}{|P(\lambda)|^2} \, d\lambda < \infty \tag{58}$$

$$\int_{-\infty}^{\infty} \frac{\log |P(\lambda)|^{-2}}{1+\lambda^2} \, d\lambda < \infty \tag{59}$$

Let $\mathcal{H}_R$ be the reproducing kernel Hilbert Space with reproducing kernel

$$R(t,t') = \int_{-\infty}^{\infty} G(t,u)\, G(t',u)\, du \tag{60}$$

where

$$G(t,u) = g(t-u), \tag{61}$$

$$g(\tau) = \int_{-\infty}^{\infty} \frac{e^{i\lambda\tau}}{P(\lambda)} \, d\lambda \tag{62}$$

$\mathcal{H}_R$ is the range of G with $L_2(-\infty,\infty)$ as domain, and has the norm

$$\| f \|^2 = \int_{-\infty}^{\infty} \frac{|\phi_f(\lambda)|^2}{|P(\lambda)|^2} \, d\lambda \tag{63}$$

where ϕ_f is the Fourier-Plancherel transform of f.

Then, if $\sum_{j=-\infty}^{\infty} f_j^2 < \infty$, the solution to the problem: Find $h \in \mathcal{H}_R$ to minimize $\|h\|^2$ subject to $h(j) = f_j$ is unique, and is obtained by solving the system (23).

7. Summary

In summary, we have proved the following

Theorem: Let $\mathcal{H}_R = \{f: f^{(r)}$ absolutely continuous on $(-\infty, \infty)$, $r = 0, 1, 2, \ldots, m-1$, $f \in L_2(-\infty, \infty)$, $L_m f \in L_2(-\infty, \infty)\}$, where $L_m f = \prod_{\nu=1}^{m} (D+\alpha_\nu) f$, $\{\alpha_\nu\}_{\nu=1}^{m}$ district positive numbers. Let $\{f_i\}_{i=-\infty}^{\infty}$ be given. Then the necessary and sufficient condition that a unique solution exists to the problem: Find $h \in \mathcal{H}_R$ such that $h(j) = f_j$, $j = \ldots -1, 0, 1, \ldots$ to minimize

$$\int_{-\infty}^{\infty} (L_m h(s))^2 \, ds$$

is that $\sum_{i=-\infty}^{\infty} f_i^2 < \infty$. In this case the solution is given by

$$h(s) = \sum_{j=-\infty}^{\infty} b_j(s) f_j \tag{64}$$

where $\{b_j(s)\}_{j=-\infty}^{\infty}$ are given by (33).

If f is any element in $\mathcal{H}_R$ satisfying $f(j) = f_j$, then

$$|f(s) - h(s)|^2 \leq \left[\int_{-\infty}^{\infty} |P(\lambda)|^2 \, |\phi_f(\lambda)|^2 \, d\lambda - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|F(\lambda)|^2}{\psi(\lambda)} \, d\lambda \right] \times \left[\int_{-\infty}^{\infty} \frac{1}{|P(\lambda)|^2} \, d\lambda - \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|\psi_s(\lambda)|^2}{\psi(\lambda)} \, d\lambda \right] \tag{65}$$

where

$$F(\lambda) = \sum_{j=-\infty}^{\infty} e^{-i\lambda j} f_j \tag{66}$$

$$\phi_f(\lambda) = \int_{-\infty}^{\infty} e^{-is\lambda} f(s)\,ds \tag{67}$$

and $P(\lambda)$, $\psi_s(\lambda)$ are given by (9), (27) and (28).

Existence and uniqueness of a solution in other Hilbert spaces has been discussed.

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