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by

Grace Wahba

Typist/ Jacquelyn R. Jones

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We generalize slightly a theorem of Shepp concerning processes equivalent to the m - 1 fold integrated Wiener process, to processes equivalent to an m - 1 fold integrated weighted Wiener process, and give the Radon-Nikodym derivative. The Fredholm determinant appearing in the R - N derivative can be evaluated, for the cases studied here, by a formula given by Kailath.

Let $X(t)$, $0 \le t \le 1$ be a zero mean Gaussian stochastic process. Let μ_o be the measure induced on path space by the covariance

$$R_o(s,t) = \int_o^1 G_o(s,u)G_o(t,u)du \tag{1}$$

$$G_o(s,u) = \frac{(s-u)_+^{m-1}}{(m-1)} c_o(u), \quad (x)+ = x,\ x > 0; \quad (x)+ = 0,\ x \le 0$$

where $c_o(u) > 0$. Since $X(t)$ has a realization under μ_o of the form

$$X(t) = \int_o^t \frac{(t-u)_+^{m-1}}{(m-1)} c_o(u)dW(u) = \int_o^t dt \int_o^{t_1} dt_1 \ldots \int_o^{t_{m-1}} c_o(u)dW(u) \tag{2}$$

where $W(u)$ is a Wiener process, we call $X(t)$ an m-1 fold integrated weighted Wiener process.

Let μ_1 be the measure induced by the covariance

$$R_1(s,t) = \int_0^1 G_1(s,u)G_1(t,u)du \tag{3}$$

If $G_1(s,u) = 0$, $s < u$, then G_1 is said to be a Volterra square root of R_1.

We have the following

Theorem:

Suppose

i) $G_1(t,u) = 0,\ t < u$

ii) $\left.\frac{\partial^j}{\partial t^j} G_1(t,u)\right|_{t\downarrow u} = 0,\quad j = 0,1,2,\ldots,m-2$

iii) $\left.\frac{\partial^{m-1}}{\partial t^{m-1}} G_1(t,u)\right|_{t\downarrow u} = c_0(u) > 0$

iv) $\left.\frac{\partial^m}{\partial t^m} G_1(t,u)\right|_{t\downarrow u} = c_1(u)$, with $\frac{c_1(u)}{c_0(u)}$ continuous

Let $M(t,u) = -\frac{1}{c_0(t)}\frac{\partial^m}{\partial t^m} G_1(t,u)$ and suppose

v) $\frac{\partial}{\partial t} M(t,u)$ exists and is bounded in t $0 \leq u \leq t \leq 1$

$\frac{\partial}{\partial u} M(t,u)$ exists and is bounded in u, $0 \leq u \leq t \leq 1$

Let M be the Hilbert-Schmidt (Volterra) operator on $L_2[0,1]$ defined by

$$(Mf)(t) = \int_0^t M(t,u)f(u)du \tag{4}$$

and let M^* be the adjoint of M.

Then μ_1 is strongly equivalent to μ_0 and

$$\frac{d\mu_1}{d\mu_0}(X) = (\det(I - M)(I - M^*))^{-\frac{1}{2}} e^{\frac{1}{2}\int_0^1\int_0^1 \frac{dX^{m-1}(s)}{c_0(s)} H(s,t) \frac{dX^{m-1}(t)}{c_0(t)}} \tag{5}$$

where $H(s,t)$ is the Hilbert-Schmidt kernel for $H = I - (I - M^*)^{-1}(I - M)^{-1}$ and

$$(\det(I - M)(I - M^*))^{-\frac{1}{2}} = e^{-\frac{1}{2}\int_0^1 \frac{c_1(u)}{c_0(u)} du} \tag{6}$$

Remark: The case $c_0(t) \equiv c_0$, a constant, is discussed in Shepp [8].

We have here the conditions that $X^{(j)}(0) = 0$, $j = 0,1,2,\dots,m-1$, and $EX(t) = 0$, under u_0 and μ_1. Shepp's study without these conditions goes through directly for c_0 not required to be a constant, and we omit discussion of these cases. We note that if i) - v) hold and $G_1(t,s)$ is a sufficiently smooth function of s, then

$$(-1)^m c_0^{\,2}(t) = \lim_{s\downarrow t} \frac{\partial^{2m-1}}{\partial s^{2m-1}} R_i(s,t) - \lim_{s\uparrow t} \frac{\partial^{2m-1}}{\partial s^{2m-1}} R_i(s,t),\ i = 0,1 \tag{7}$$

If $G_1(s,t)$ is a Green's function for a differential operator of the form

$$(L_1 f) = \sum_{j=0}^{m} a_{m-j}(t) D^j f(t) \tag{8}$$

then

$$c_0(t) = \frac{1}{a_0(t)} \tag{9}$$

$$c_1(t) = \frac{a_1(t)}{a_0^2(t)} \tag{10}$$

The situation (9) covers the processes considered in [10].

An outline of the proof of strong equivalence goes as follows. Letting G_i, $i = 0,2$ be the (Hilbert-Schmidt) operators defined by

$$(G_i f)(t) = \int_0^t G_i(t,u)f(u) \tag{11}$$

we note that

$$(G_0(Mf))(t) = \int_0^t \frac{(t-x)_+^{m-1}}{(m-1)!} \int_0^x \frac{\partial^m}{\partial x^m} G_1(x,u)f(u) \tag{12}$$

$$= \int_0^t du\ f(u) \int_u^t \frac{(t-x)_+^{m-1}}{(m-1)!} \frac{\partial^m}{\partial x^m} G_1(x,u)dx \tag{13}$$

and since

$$G_1(t,u) = \frac{(t-u)_+^{m-1}}{(m-1)!} c_0(u) + \int_u^t \frac{(t-x)_+^{m-1}}{(m-1)!} \frac{\partial^m}{\partial x^m} G_1(x,u)dx \tag{14}$$

we have

$$G_1 = G_0(I - M). \tag{15}$$

We may write

$$R_i = G_i G_i^* \quad , i = 0,1 \tag{16}$$

where G_i^* is the adjoint operator to G_i.

Let $\{\alpha_i\}_{i=1}^{\infty}$ and $\{\phi_i(t)\}_{i=1}^{\infty}$ be the eigenvalues and orthonormalized eigenfunctions of R_0 and let

$$R_0^{-1/2} f = \sum_{i=1}^{\infty} \frac{1}{\sqrt{\alpha_i}} (f,\phi_i)\phi_i \tag{17}$$

whenever $\sum_{i=1}^{\infty} \frac{1}{\alpha_i} (f,\phi_i)^2 < \infty$. A version of the Hajek-Feldman theorem stated in Root [8] says that μ_1 and μ_0 are equivalent if and only if

$$R_0^{-1/2} R_1 R_0^{-1/2} = I - B \tag{18}$$

where B is Hilbert-Schmidt and I - B is invertible. Using the terminology of Hajek, μ_1 and μ_0 are strongly equivalent if B is of trace class and then det(I - B) exists. Now $R_0^{-1/2} R_1 R_0^{-1/2}$ is unitarily equivalent to

$$G_0^{-1} G_1 G_1^{*} G_0^{*-1} = (I - M)(I - M^*) = I - (M + M^* - MM^*) \tag{19}$$

Since M is Volterra, (I - M) and hence (I - M)(I - M*) are invertible. See, for example, Petrovskii, [7]. Thus, 0 is not in the spectrum of (I - M)(I - M*) and -1 is not in the spectrum of K = (I - M)(I - M*) - I. K is obviously Hilbert-Schmidt, and hence $\mu_1 \equiv \mu_0$. Assumption v) insures that M + M* and hence M + M* - MM*, and B are of trace class, as follows. Upon integrating by parts,

$$((M + M^*)f)(t) = \int_0^t M(t,s)f(s)ds + \int_t^1 M(s,t)f(s)ds$$

$$= M(1,t) \int_0^1 f(s)ds + \int_0^1 C(t,u)f(u) \tag{20}$$

where

$$C(t,u) = \int_0^1 A(t,s)I_s(u)ds$$

with

$$I_s(u) = 1 \qquad s \geq u \qquad (21)$$
$$= 0 \qquad s < u$$

and

$$A(t,s) = -\frac{\partial}{\partial s} M(t,s) \qquad t > s \qquad (22)$$
$$= -\frac{\partial}{\partial s} M(s,t) \qquad t < s$$

Since $M + M^*$ has a representation as the sum of a rank 1 operator plus the product of two Hilbert-Schmidt operators, it is of trace class.

$I - (I - M)(I - M^*)$ of trace class insures that

$$H = I - (I - M^*)^{-1}(I - M)^{-1}$$

is also of trace class. Let

$$h = -M(I - M)^{-1} \qquad (23)$$

that is,

$$(I - h) = (I - M)^{-1} \qquad (24)$$

then

$$H = h + h^* + hh^* \qquad (25)$$

and h is given by the convergent Neuman series [1]

$$h(t,u) = -\sum_{j=1}^{\infty} M^j(t,u) \qquad (26)$$

$$M^1(t,u) = M(t,u)$$

$$M^{k+1}(t,u) = \int_u^t M^k(t,\xi)M(\xi,u)d\xi$$

The (Fredholm) determinant (6) may be evaluated by noting that

$$\det(I - M)(I - M^*) = \det(I - H)^{-1}, \tag{27}$$

and then using the following formula, given in Kailath [6].

$$\det(I - H)^{-1} = e^{\operatorname{tr}(h+h^*)} \tag{28}$$

Equation (6) then follows from

$$\operatorname{tr}(h+h^*) = \int_0^1 (h+h^*)(t,t)dt \tag{29}$$

and

$$(h+h^*)(t,t) = -(M + M^*)(t,t) = \frac{c_1(t)}{c_0(t)} \ . \tag{30}$$

If $M(t,u) = M(t - u)$, then $h(t,u)$ may frequently be found explicitly by operational methods (see Erdelyi [2]).

We briefly outline a proof of (5). Missing details can be filled in from Sections 10 and 12 of [9]. The only substantial difference between the argument here and that of Shepp is due to the fact that we take into account the fact that the eigenfunctions of R_0 and R_1 are not necessarily the same. The idea behind the algebra below is the simultaneous diagonalization of two covariance functions. (See [4]).

Let

$$X_n(t) = \sum_{j=1}^{n} V_j \sqrt{\alpha_j}\, \phi_j(t) \tag{31}$$

where

$$V_j = \frac{1}{\sqrt{\alpha_j}} \int_0^1 X(s)\phi_j(s)ds \tag{32}$$

Under μ_0, (31) is the nth partial sum of the Karhunen-Loeve expansion of $X(t)$ and under either μ_0 or μ_1 $X_n(t)$ converges in quadratic mean to $X(t)$.

Now note that ϕ_j, $j = 1,2,\ldots$ is in the domain of $G_0^{-1} = L_0$, where $(L_0\, f)(t) = \frac{1}{c_0(t)} f^{(m)}(t)$, since $\phi_j \in \mathcal{H}_{R_0}$, the reproducing kernel Hilbert space with reproducing kernel R_0.

Consider

$$Z_n = (G_0^{-1}X_n(t), HG_0^{-1}X_n(t)) = \int_0^1\int_0^1 \frac{dX_n^{(m-1)}(s)}{c_0(s)} H(s,t)\frac{dX_n^{(m-1)}(t)}{c_0(t)}$$

$$= \sum_{i,j=0}^{n} V_i(\xi_i, H\xi_j)V_j \tag{33}$$

where

$$\xi_i = G_0^{-1}\sqrt{\alpha_i}\, \phi_i \tag{34}$$

and

$$(\xi_i,\xi_j) = (G_0^{-1}\sqrt{\alpha_i}\,\phi_i,\ G_0^{-1}\sqrt{\alpha_j}\,\phi_j) = (\sqrt{\alpha_i}\,\phi_i,\ R_0^{-1}\sqrt{\alpha_j}\,\phi_j) = 1,\ i = j$$
$$= 0,\ i \neq j \qquad (35)$$

We show that the q.m. limit of $\frac{1}{2} Z_n$ is the exponent in $\frac{d\mu_1}{d\mu_0}$.

Let $\{\gamma_i\}_{i=1}^{\infty}$ be a complete orthonomal set of eigenfunctions for the trace class symmetric operator $H = (I - M)^{-1}(I - M^*)^{-1} - I$ and denote the eigenvalues of H by $\{(1 - \frac{1}{\sigma_i})\}_{i=1}^{\infty}$, where we know that $\sum_{i=1}^{\infty} |1 - \frac{1}{\sigma_i}| < \infty$. (If H has a null space, complete the eigenfunctions in any manner). Let

$$\theta_{kn} = \sum_{i=1}^{n} (V_i\xi_i,\gamma_k) \qquad k,n = 1,2,\ldots \qquad (36)$$

Then

$$\int_0^1\int_0^1 \frac{dX_n^{(m-1)}(s)}{c_0(s)} H(s,t) \frac{dX_n^{(m-1)}(t)}{c_0(t)} = \sum_{i,j=1}^{n} V_i(\xi_i,H\xi_j)V_j$$

$$= \sum_{i,j=1}^{n} V_iV_j \sum_{k=1}^{\infty} (\xi_i,\gamma_k)(\xi_j,\gamma_k)(1-\frac{1}{\sigma_k})$$

$$= \sum_{k=1}^{\infty} \theta_{kn}^2 (1 - \frac{1}{\sigma_k}) \qquad (37)$$

Letting E_i be expectation under μ_i, $i = 0,1$, we have

$$E_0 V_iV_j = 1, \quad i = j$$
$$= 0, \quad i \neq j$$

and hence

$$E_0\theta_{kn}\theta_{\ell n} = E_0 \sum_{i=1}^{n} (V_i\xi_i,\gamma_k) \sum_{j=1}^{n} (V_j\xi_j,\gamma_\ell) \qquad (38)$$

$$= \sum_{\nu=1}^{n} (\xi_\nu,\gamma_k)(\xi_\nu,\gamma_\ell)$$

$$= (P_n\gamma_k,\ P_n\gamma_\ell)$$

where P_n is the projection operator in $\mathcal{L}_2$ onto the n-dimensional space spanned by $\{\xi_\nu\}_{\nu=1}^n$. It then follows that θ_{kn}, n = 1,2,... is a Cauchy sequence for each k,

$$\theta_{kn} \xrightarrow{q.m.,\mu_0} \theta_k$$

with

$$E\theta_k\theta_\ell = \delta_{k\ell} \tag{39}$$

Now,

$$\begin{aligned} E_1 V_i V_j &= \frac{1}{\sqrt{\alpha_i}} \frac{1}{\sqrt{\alpha_j}} (\phi_i, R_1\phi_j) \\ &= \frac{1}{\sqrt{\alpha_i}} \frac{1}{\sqrt{\alpha_j}} (\phi_i, G_0(I-M)(I-M^*)G_0^*\phi_j) \\ &= (\frac{1}{\sqrt{\alpha_i}} G_0^*\phi_i, (I-M)(I-M^*)\frac{1}{\sqrt{\alpha_j}} G_0^*\phi_j) \\ &= (\xi_i, (I-M)(I-M^*)\xi_j) \end{aligned} \tag{40}$$

since

$$\sqrt{\alpha_i}\, G_0^{-1}\phi_i = \frac{1}{\sqrt{\alpha_i}} G_0^{*-1}\phi_i = \xi_i. \tag{41}$$

Thus

$$\begin{aligned} E_1\theta_{kn}\theta_{\ell n} &= \sum_{i=1}^{n}\sum_{j=1}^{n} (\xi_i,\gamma_k)(\xi_j,\gamma_\ell)(\xi_i,(I-M)(I-M^*)\xi_j) \\ &= (P_n\gamma_k, (I-M)(I-M^*)P_n\gamma_\ell) \end{aligned} \tag{42}$$

Since $(I-M)(I-M^*)$ is bounded, and $(I-M)(I-M^*)\gamma_\ell = \sigma_\ell$, $\ell = 1,2,\ldots$ it then will follow that for each k,

$$\theta_{kn} \xrightarrow{q.m.,\mu_1} \theta_k$$

with

$$E\theta_k\theta_\ell = \delta_{k\ell}\sigma_\ell \tag{43}$$

The $\{\theta_\ell\}_{\ell=1}^{\infty}$ are complete in the Hilbert space spanned by $X(t)$, $0\le t\le 1$, and

$$Z_n \xrightarrow{q.m.,\mu_0 \text{ and } \mu_1} \sum_{k=1}^{\infty} \theta_k^{\,2}(1-\frac{1}{\sigma_k}) = \int_0^1\int_0^1 \frac{dX^{(m)}(s)}{c_0(s)} H(s,t) \frac{dX^{(m)}(t)}{c_0(t)} . \tag{44}$$

It will then follow that

$$\frac{d\mu_1}{d\mu_0} = \frac{1}{(\prod_{j=1}^{\infty}\sigma_j)^{1/2}} e^{\frac{1}{2}\sum_{k=1}^{\infty}\theta_k^{\,2}(1-\frac{1}{\sigma_k})} \tag{45}$$

$$=(\det(I-M)(I-M^*))^{-1/2} e^{\frac{1}{2}\int_0^1\int_0^1 \frac{dX^{(m)}(s)}{c_0(s)} H(s,t) \frac{dX^{(m)}(t)}{c_0(t)}} . \tag{46}$$

This ends the discussion of the proof of the theorem.

Suppose, on the other hand, that R_1 is an arbitrary Hilbert-Schmidt operator with a representation

$$R_1 = G_0(I+K)G_0^{\,*}. \tag{47}$$

Then, the necessary and sufficient conditions for μ_1 to be strongly equivalent to μ_0 is that K be a trace class Hilbert-Schmidt operator with -1 not in the spectrum of K. If, say, K has a continuous kernel $K(s,t)$, then

$$\left.\frac{\partial^{i+j}}{\partial s^i \partial t^j} R_1(s,t)\right|_{s=t=0} = 0 \qquad i,j = 0,1,2,\ldots\ m-1 \tag{48}$$

and

$$\frac{1}{c_0(s)c_0(t)} \frac{\partial^{2m}}{\partial s^m \partial t^m} R_1(s,t) = K(s,t), \qquad s \neq t, \tag{49}$$

as in Shepp where c_0 is taken to be a constant. The discussion concerning Z_{1n} used only the assumption that $H = I - (I + K)^{-1}$ is of trace class, so that, if μ_1 is strongly equivalent to μ_0, then (5) with the substitution (27) always gives the likelihood function.

It is known that if, for example $K(s,t)$ is continuous and -1 is not an eigenvalue, then there always exists a Volterra operator h, with $h(s,t)$ continuous for $0 \leq t < s$, satisfying (25). See [5] and references cited there. Thus, if the right hand side of (7) is well defined, then the left hand side must give $c_0(t)$. Further, there exists an M solving (24).

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