DEPARTMENT OF STATISTICS

University of Wisconsin
Madison, Wisconsin 53706

Technical Report No. 282

September 1971

ON THE MINIMIZATION OF A QUADRATIC FUNCTIONAL SUBJECT TO A CONTINUOUS FAMILY OF LINEAR INEQUALITY CONSTRAINTS

by

Grace Wahba

Typist: Bernice R. Weitzel

This work was supported by the Wisconsin Alumni Research Foundation and by the Air Force Office of Scientific Research under Grant No. AF-AFOSR 69-1803.

ON THE MINIMIZATION OF A QUADRATIC FUNCTIONAL SUBJECT TO A CONTINUOUS FAMILY OF LINEAR INEQUALITY CONSTRAINTS

by

Grace Wahba[†]

[†]Department of Statistics, University of Wisconsin, Madison, Wisconsin 53706.

This work was supported by the Wisconsin Alumni Research Foundation.

Abstract

The problem of minimizing a positive definite quadratic functional subject to a continuous family of linear inequality constraints is studied. Upper and lower bounds are given for the value of the functional at the minimum. In certain cases, the given bounds coincide, and an explicit formula for the solution is given. Convergence rates for a sequence of (computable) approximate solutions obtained by discretizing the constraint set are established.

ON THE MINIMIZATION OF A QUADRATIC FUNCTIONAL SUBJECT TO A CONTINUOUS FAMILY OF LINEAR INEQUALITY CONSTRAINTS

by

Grace Wahba

1. Introduction. Ritter, in [6] considers the problem of minimizing a quadratic functional $J(f)$ subject to a finite set of linear inequalities of the form

$$\alpha(t) \leq \langle \eta_t, f \rangle_{\mathcal{H}} \leq \beta(t) , \quad t \varepsilon T_n . \tag{1.1}$$

Here $f \varepsilon \mathcal{H}$, a Hilbert space, $T_n = \{t_1, t_2, \ldots, t_n\}$, a finite (discrete) index set, $\{\eta_t, t \varepsilon T_n\}$ is a set of elements of $\mathcal{H}$, $\langle \cdot,\cdot \rangle_{\mathcal{H}}$ is the inner product in $\mathcal{H}$ and J is a non-negative definite quadratic functional on $\mathcal{H}$ with null space of dimension $m < n$. The Hilbert spaces considered are such that the linear functionals N_t^{ν} defined by $N_t^{\nu} f \to f^{(\nu)}(t)$, $t \varepsilon T_n$, $\nu = 0, 1, 2, \ldots, m-1$, are continuous, and $f^{(m)} \varepsilon \mathcal{L}_2$ The $\{\eta_t, t \varepsilon T_n\}$ are representers of linear combinations of such continuous linear functionals. Thus the constraints become

$$\alpha(t) \leq \sum_{j=0}^{m-1} c_j(t) \, f^{(j)}(t) \leq \beta(t) , \quad t \varepsilon T_n, \tag{1.2}$$

for some real numbers $\{c_j(t), j = 0, 1, \ldots, m-1, t \varepsilon T_n\}$. $J(f)$ is of the form

$$J(f) = \int_0^1 (L_m f(t))^2 dt \tag{1.3a}$$

where

$$L_m f(t) = \sum_{j=0}^{m} a_j(t)\, f^{(j)}(t) \ , \ a_m(t) \neq 0 \ \text{ in } [0,1] \ . \tag{1.3b}$$

It was shown in [6] that the solution to this problem can be reduced to the solution of a (finite) standard quadratic programming problem.

Mangasarian and Schumaker [5], and Laurent [4] considered generalizations of this problem obtained by enlarging the constraint set by replacing T_n by $T = D \cup E$, where D is a finite set of points and E is a finite union of closed, bounded intervals. An example of such a set of constraints is

$$\begin{aligned} &\alpha_D(t) \leq \sum_{j=0}^{m-1} c_j(t)\, f^{(j)}(t) \leq \beta_D(t) \ , \quad t \varepsilon D = \{\xi_1, \xi_2, \ldots, \xi_\ell\} \ . \\ &\alpha_{E_i}(t) \leq \sum_{j=0}^{m-1} b_{ij}(t)\, f^{(j)}(t) \leq \beta_{E_i}(t) \ , \quad t \varepsilon E_i = [0,1],\ E = \bigcup_i E_i \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \ldots, k. \end{aligned} \tag{1.4}$$

Various characterizations of the (possibly non-unique) solutions are discussed. Daniel [2] considered an approximate method of solving the minimization problem with J and the constraints as in (1.3) and (1.4) Daniel's procedure consists of replacing E_i in (1.4) by the discrete set $E_{in} \subset E_i$,

$$E_{in} = \{\tfrac{j}{n},\ j = 0, 1, 2, \ldots, n\} \ .$$

A solution f_n^* to the problem: minimize $J(f)$ of (1.3) subject to

$$\alpha_D(t) \le \sum_{j=0}^{m-1} c_j(t)\, f^{(j)}(t) \le \beta_D(t)\ , \quad t\varepsilon D$$
$$\alpha_{E_i}(t) \le \sum_{j=0}^{m-1} b_{ij}(t)\, f^{(j)}(t) \le \beta_{E_i}(t), \quad t\varepsilon E_{in} \tag{1.5}$$

may be found by the methods described in [6], and is taken by Daniels as the nth approximation to the solution of the original minimization problem with constraints (1.4). He discusses the convergence properties f_n^* as $n \to \infty$. No convergence rates are given however.

The main result of this paper is the establishment of convergence rates for approximate solutions obtained by discretizing the constraints. The minimization problem we consider is more specialized in one sense and more general in another, than the one considered by Daniel.

To ease the analysis somewhat, in this note we assume that $J(f)$ is strictly positive definite over $\mathcal{H}$, which is otherwise an arbitrary separable Hilbert space. Then, without further loss of generality we may let

$$||f||_{\mathcal{H}}^2 = J(f)\ . \tag{1.6}$$

and $2\left\langle f_1, f_2\right\rangle_{\mathcal{H}} = ||f_1 + f_2||_{\mathcal{H}}^2 - ||f_1||_{\mathcal{H}}^2 - ||f_2||_{\mathcal{H}}^2$

As an example, let $\mathcal{H}$ be the (Sobolev) space $W^{m,2}$,

$$W^{m,2} = \{f:\ f^{(m-1)} \text{ absolutely continuous },\ f^{(m)}\varepsilon\mathcal{L}_2[0,1]\} \tag{1.7}$$

and

$$J(f) = \sum_{j=0}^{m-1} \lambda_j (f^{(j)}(0))^2 + \int_0^1 (L_m f(t))^2 dt \tag{1.8}$$

where $\lambda_i > 0$, $i = 0, 1, \ldots, m-1$, and L_m is as in (1.3b). We also assume that the constraints are of the form

$$0 < \alpha(t) \leq \langle \eta_t, f \rangle_{\mathcal{H}} \quad , \quad t \varepsilon T \ , \tag{1.9}$$

$T = D \cup E$, where D is a finite set of points and $E = [a,b]$ is a closed, bounded interval. The results go through for E a finite union of closed, bounded intervals, we omit the details. The family $\{\eta_t, t \varepsilon T\}$ is an arbitrary family in $\mathcal{H}$ except that we assume that the kernel $Q(s,t)$ defined on $T \times T$ by

$$Q(s,t) = \frac{1}{\alpha(s)\alpha(t)} \langle \eta_s, \eta_t \rangle_{\mathcal{H}} \quad , \quad s,\ t \varepsilon T \tag{1.10}$$

is strictly positive definite on $T \times T$ (i.e., $\{\eta_t, t \varepsilon T\}$ are linearly independent). These conditions will insure that there is a unique solution $f^* \varepsilon \mathcal{H}$ to the problem:

$$\text{minimize} \quad J(f) \qquad (= ||f||^2_{\mathcal{H}})$$

subject to

$$\alpha(t) \leq \langle \eta_t, f \rangle_{\mathcal{H}} \quad , \quad t \varepsilon T \ . \tag{1.11}$$

In this note we reduce the study of the solution f^* and its approximants f_n^* to the study of properties of the kernel Q. Section 2 has some preliminaries. In Section 3 we find upper and lower bounds on $||f^*||_{\mathcal{H}}$ in terms of Q when Q is positive. When the upper and lower bounds given coincide we may determine f^* by inspection. This easy result is summarized in

Theorem 1

If $Q(s,t) \geq 0$, $s, t \in T$, then

i) $$\inf_{s\in T} \sup_{t\in T} \frac{\sqrt{Q(s,s)}}{Q(s,t)} \geq \|f^*\|_{\mathcal{H}} \geq \sup_{t\in T} \frac{1}{\sqrt{Q(t,t)}} \tag{1.12}$$

ii) If there exists a (necessarily unique) $s_* \in T$ for which

$$\sup_{t\in T} \frac{1}{\sqrt{Q(t,t)}} = \frac{1}{\sqrt{Q(s_*,s_*)}} \tag{1.13}$$

and

$$Q(s_*,s_*) = \inf_t Q(s_*,t) \tag{1.14}$$

then

$$f^* = \frac{1}{\alpha(s_*)Q(s_*,s_*)} \eta_{s_*} \tag{1.15}$$

In Section 4 we find rates of convergence of a sequence of approximate solutions $\{f_n^*\}$, to f^* in terms of the continuity properties of Q on $E = [a,b]$. More precisely, let f_n^* be the (unique) solution to the problem:

$$\text{minimize } J(f)$$

subject to

$$\alpha(t) \leq \langle \eta_t, f \rangle_{\mathcal{H}}, \quad t \in T_n \tag{1.16}$$

where

$$T_n = D \cup E_n, \tag{1.17}$$

$$E_n = \{a \leq t_1 < t_2 < \ldots < t_n = b\}$$

f_n^* may be found by solving a standard quadratic programming problem (see Section 4 for details).

Let

$$\max_{i = 1, 2, \ldots\, n-1} |t_{i+1} - t_i| = \Delta \tag{1.18}$$

Then we prove the following:

Theorem 2

Let $Q(s,t)$ have continuous mixed partial derivatives to order $2p - 2$ and bounded $2p - 1$ st order derivatives on $E \times E$. Then

$$||f_n^* - f^*||_{\mathcal{H}} = O(\Delta^{\frac{1}{2}\min(p - \frac{1}{2}, 2)}) \tag{1.19}$$

If the evaluation functionals $N_s f \rightarrow f(s)$, and $N_s^\nu f \rightarrow f^{(\nu)}(s)$, $\nu = 1, \ldots, m-1$ are all continuous in $\mathcal{H}$, as is true in most of the interesting examples, then (1.19) entails pointwise convergence of $f_n^{*(\nu)}(s)$ to $f^{*(\nu)}(s)$ for $\nu = 0, 1, \ldots, m-1$, as follows: If R_s^ν is the representer of N_s^ν, that is,

$$N_s^\nu f = \left\langle R_s^\nu, f \right\rangle_{\mathcal{H}} = f^{(\nu)}(s) \;, \; f \varepsilon \mathcal{H}, \quad \nu = 0, 1, \ldots\; m-1, \quad N_s^0 = N_s \tag{1.20}$$

then

$$\begin{aligned} |f_n^{*(\nu)}(s) - f^{*(\nu)}(s)| &= | \left\langle f_n^* - f^* \,,\, R_s^\nu \right\rangle_{\mathcal{H}} | \\ &\leq ||R_s^\nu||_{\mathcal{H}} ||f_n^* - f^*||_{\mathcal{H}}. \end{aligned} \tag{1.21}$$

An as example of the application of Theorem 2 to a problem similar to that considered by Daniel, let $J(f)$ be given by (1.8), where the $\{a_j\}_{j=0}^{m}$ appearing in (1.3b) are of continuity class C^{2m}. Then we may take $\mathcal{H}$ to be $W^{m,2}$ of (1.7) with the norm defined by (1.6) and (1.8). In this case the linear functionals $N_t^{\nu}f \to f^{(\nu)}(t)$ are all continuous for $\nu = 0, 1, \ldots, m-1$, $t\varepsilon[0,1]$.

Let the constraints involving E be of the form

$$\alpha(t) \leq M_t f \overset{\text{def}}{=} \sum_{j=0}^{q} b_j(t) f^{(j)}(t), \quad t\varepsilon[a,b] \tag{1.22}$$

where $q \leq m-1$. Since the evaluation functionals $N_t f \to f(t)$ are all continuous in $W^{m,2}$ with the norm defined by (1.6) and (1.8), $W^{m,2}$ possess a reproducing kernel, call it $R(s,t)$.

By the general properties of reproducing kernels, we may always find $\langle \eta_s, \eta_t \rangle_{\mathcal{H}}$ of (1.10), where η_s is the representer of an arbitrary continuous linear functional M_s, if the reproducing kernel $R(s,t)$ is known. The relationship is

$$\langle \eta_s, \eta_t \rangle_{\mathcal{H}} = M_{s(u)} M_{t(v)} R(u,v) \tag{1.23}$$

where $M_{s(u)}$ means the linear functional M_s applied to R considered as a function of u. (For further discussion of Hilbert spaces possessing a reproducing kernel, see [7] and references cited there.

It is well known (see, for example Kimeldorf and Wahba [3]) that $W^{m,2}$ with the norm defined by (1.6) and (1.8) possess the reproducing kernel $R(s,t)$ given by

$$R(s,t) = \sum_{j=0}^{m-1} \frac{\phi_j(s)\phi_j(t)}{\lambda_j} + \int_0^{\min(s,t)} G(s,u)\, G(t,u)du \qquad s,\ t\varepsilon[0,1] \tag{1.24}$$

where

$$L_m \phi_j = 0 \qquad j = 0,\ 1,\ 2,\ \ldots,\ m-1$$

$$\phi_j^{(\nu)}(0) = \delta_{\nu j} \qquad \nu,\ j, = 0,\ 1,\ 2,\ \ldots,\ m-1\ .$$

and $G(s,u)$ is the Green's function for the problem

$$L_m f = g$$

$$f^{(\nu)}(0) = 0 \quad , \nu = 0,\ 1,\ 2,\ \ldots,\ m-1\ .$$

Then

$$\begin{aligned} Q(s,t) &= \frac{1}{\alpha(s)\alpha(t)} \left\langle \eta_s, \eta_t \right\rangle_{\mathcal{H}} = \frac{1}{\alpha(s)\alpha(t)} M_{s(u)} M_{t(v)} R(u,v) \\ &= \frac{1}{\alpha(s)\alpha(t)} \sum_{j,k=0}^{q} b_j(s) b_k(t) \frac{\partial^{j+k}}{\partial s^j \partial t^k} R(s,t)\ , \qquad s,\ t\varepsilon[a,b] \end{aligned} \tag{1.25}$$

By recalling the properties of Green's functions, we see that

$$\frac{\partial^{j+k}}{\partial s^j \partial t^k} R(s,t)\ , \qquad j,\ k = 0,\ 1,\ \ldots,\ q,\ s,\ t\varepsilon[a,b] \tag{1.26}$$

has continuous mixed partial derivatives to order at least $2(m-q)-2$ and bounded (left and right) derivatives of order $2(m-q)-1$. Thus if α, $b_j \varepsilon C^{2(m-q)-2}$, $\alpha^{2(m-q)-1}$, $b_j^{2(m-q)-1}$ bounded, then Q of (1.25) satisfies the hypotheses of Theorem 2 with $p = m-q$.

2. Transformation of the problem to canonical form. Let $\mathcal{H}$ be a Hilbert space, let T be as in (1.9) and let $\{\eta_t, t\varepsilon T\}$ be a family of linearly independent elements of $\mathcal{H}$. Let V be the span of $\{\eta_t, t\varepsilon T\}$ in $\mathcal{H}$. We seek to find $f\varepsilon\mathcal{H}$ to

$$\text{minimize } ||f||^2_{\mathcal{H}} \tag{2.1}$$

subject to

$$0 < \varepsilon \leq \alpha(t) \leq \langle \eta_t, f \rangle_{\mathcal{H}} , \quad t\varepsilon T . \tag{2.2}$$

For any $f\varepsilon\mathcal{H}$, let $f = f_1 + f_2$ where $f_1\varepsilon V$ and $f_2\varepsilon V^{\perp}$. Since $\langle \eta_t, f_2 \rangle_{\mathcal{H}} = 0$, $t\varepsilon T$, then f satisfies (2.2) if f_1 satisfies (2.2) and it is obvious any solution to this problem will be in V. Let

$$Q(s,t) = \frac{1}{\alpha(s)\alpha(t)} \langle \eta_s, \eta_t \rangle_{\mathcal{H}} , \quad s, t\varepsilon T \tag{2.3}$$

Then Q is a symmetric non negative definite kernel on $T\times T$, and we assume that $Q(t,t) = ||\eta_t||^2_{\mathcal{H}}$ is strictly greater than 0 for all $t\varepsilon T$. Thus there exists a unique reproducing kernel Hilbert space $\mathcal{H}_Q$ of functions defined on T with reproducing kernel Q. Denote the inner product in $\mathcal{H}_Q$ by $\langle \cdot,\cdot \rangle_Q$. $\mathcal{H}_Q$ has the usual properties that the function Q_t defined by

$$Q_t(s) = Q(t,s) \qquad s\varepsilon T \tag{2.4}$$

satisfies

$$Q_t \ \varepsilon \ \mathcal{H}_Q , \ \forall \ t\varepsilon T \tag{2.5}$$

$$\left\langle Q_t, g \right\rangle_Q = g(t) , \quad \forall \ g \in \mathcal{H}_Q , \ t \in T . \tag{2.6}$$

and the family $\{Q_t, t\varepsilon T\}$ span $\mathcal{H}_Q$. There is an isometric isomorphism between V and $\mathcal{H}_Q$ generated by the correspondence

$$\frac{1}{\alpha(t)} \eta_t \varepsilon V \sim Q_t \varepsilon \mathcal{H}_Q . \tag{2.7}$$

This follows since $\{\eta_t, t\varepsilon T\}$ span V, and

$$\frac{1}{\alpha(s)\alpha(t)} \left\langle \eta_s, \eta_t \right\rangle_{\mathcal{H}} = Q(s,t) = \left\langle Q_s, Q_t \right\rangle_Q . \tag{2.8}$$

Furthermore, $f\varepsilon V \sim g\varepsilon \mathcal{H}_Q$ iff

$$\frac{1}{\alpha(t)} \left\langle \eta_t, f \right\rangle_{\mathcal{H}} = g(t) = \left\langle Q_t, g \right\rangle_Q , \quad t\varepsilon T . \tag{2.9}$$

If $f\varepsilon V \sim g\varepsilon \mathcal{H}_Q$, then

$$||f||_{\mathcal{H}} = ||g||_Q . \tag{2.10}$$

Thus the problem may now be reformulated as: Find $g\varepsilon \mathcal{H}_Q$ to

$$\text{minimize} \quad ||g||_Q^2 \tag{2.11a}$$

subject to

$$1 \leq \left\langle Q_t, g \right\rangle_Q = g(t) , \quad t\varepsilon T \tag{2.11b}$$

Then, if g^* is a solution to this problem, a solution f^* to the problem of (2.1) and (2.2) is given by finding f^* which corresponds to g^* under the isomorphism "~" of (2.9).

$$\frac{1}{\alpha(t)} \left\langle \eta_t, f^* \right\rangle_{\mathcal{H}} = g^*(t) , \quad t \varepsilon T \tag{2.12}$$

In many cases (2. 2) may be solved analytically for f^*, given g^* by noting that if

$$g^*(t) = \sum_{j=1}^{\ell} c_j Q_{t_j}(t) , \tag{2.13}$$

for some constants $\{c_j\}_{j=1}^{\ell}$, then

$$f^*(t) = \sum_{\ell=1}^{\ell} c_j \; \eta_{t_j}(t)/\alpha(t_j) \tag{2.14}$$

and, if

$$g^*(t) = \lim_{\ell \to \infty} \sum_{j=1}^{\ell} c_{j\ell} \; Q_{t_{j\ell}}(t) \tag{2.15}$$

then

$$f^*(t) = \lim_{\ell \to \infty} \sum_{j=1}^{\ell} c_{j\ell} \; \eta_{t_{j\ell}}(t)/\alpha(t_{j\ell}) . \tag{2.16}$$

3. A class of problems for which the solution may be obtained by inspection.

Let

$$S = \{\psi\colon \psi \varepsilon \mathcal{H}_Q \;,\; ||\psi|| = 1 \;,\; \psi(t) \geq 0\} \tag{3.1}$$

S is closed because it is the intersection of the boundary of the unit ball and the closed hyperplanes $\{\psi\colon \langle \psi, Q_t \rangle_Q \geq 0,\ t\varepsilon T\}$. Define

$$\Lambda_\psi = \sup_t \frac{1}{\psi(t)} \,. \tag{3.2}$$

Then, $g\varepsilon \mathcal{H}_Q$ and $g(t) \geq 1$ entails that

$$g = (\Lambda_\psi + \delta)\psi \tag{3.3}$$

for some $\psi \varepsilon S$, and $\delta \geq 0$. Then

$$||g||_Q = (\Lambda_\psi + \delta) \tag{3.4}$$

and it is obvious that the solution to the minimization problem of (2.11) is of the form

$$g = \Lambda_\psi \psi \tag{3.5}$$

for some $\psi \varepsilon S$, with $||g||_Q = \Lambda_\psi$. Then the problem becomes: Find $\psi \varepsilon S$ for which Λ_ψ is a minimum, that is, find ψ^* for which

$$\inf_{\psi\varepsilon S}\ \sup_{t\varepsilon T} \frac{1}{\psi(t)} = \sup_{t\varepsilon T} \frac{1}{\psi^*(t)} \tag{3.6}$$

This may be recognized as a problem in game theory.

Player I chooses $t\varepsilon T$, player II chooses $\psi\varepsilon S$ and the payoff to player I is $\frac{1}{\psi(t)}$. Here we are trying to find a good strategy ψ^* for player II. (See, e.g. [1]).

The following lemma will be useful in the sequel.

Lemma 1

Let Λ be defined by

$$\Lambda = \inf_{\psi\varepsilon S}\ \sup_{t\varepsilon T} \frac{1}{\psi(t)} \tag{3.7}$$

and suppose

$$\inf_t \psi_i(t) = \frac{1}{\Lambda+\varepsilon_i}, \quad i = 1, 2, \tag{3.8}$$

where ψ_i, $i = 1, 2$ εS and (of necessity) $\varepsilon_i \geq 0$. Let $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$ Then

$$\left\langle \psi_1, \psi_2 \right\rangle_Q \geq 1 - 4\frac{\varepsilon}{\Lambda} \tag{3.9}$$

Proof. By definition of Λ, and the fact that $\frac{(\psi_1(t)+\psi_2(t))/2}{||(\psi_1+\psi_2)/2||_Q}$ εS,

$$\frac{1}{\Lambda} \geq \inf_t \frac{(\psi_1(t)+\psi_2(t))/2}{||(\psi_1+\psi_2)/2||_Q} \geq \frac{1}{(\Lambda+\varepsilon)}\ \frac{1}{||(\psi_1+\psi_2)/2||_Q} = \frac{1}{(\Lambda+\varepsilon)(\frac{1}{2}+\frac{1}{2}\left\langle \psi_1, \psi_2 \right\rangle_Q)^{\frac{1}{2}}} \tag{3.10}$$

giving

$$\left\langle \psi_1, \psi_2 \right\rangle_Q \geq 2(1/(1+\varepsilon/\Lambda))^2 - 1 \geq 1 - 4(\varepsilon/\Lambda) \ . \tag{3.11}$$

Lemma 1 shows that if $\{\psi_n\}$ is a sequence of elements in S with

$$\Lambda_{\psi_n} = \sup_t \frac{1}{\psi_n(t)} \downarrow \Lambda \ , \tag{3.12}$$

then $\{\psi_n\}$ is a Cauchy sequence in the closed set S. Thus a solution ψ^* to the problem of (3.6) always exists and is unique, as is well known. Furthermore $\Lambda = \Lambda_{\psi^*}$; and $g^* = \Lambda_{\psi^*}\psi^*$ with $||g^*||_Q = \Lambda$, is the unique solution to the problem of (2.11). The unique element $f^* \varepsilon V$ satisfying

$$\frac{1}{\alpha(t)} \left\langle \eta_t, f^* \right\rangle_{\mathcal{H}} = g^*(t) \tag{3.13}$$

is the solution to the problem of (2.1) and (2.2) and

$$||g^*||_Q = ||f^*||_{\mathcal{H}} = (J(f^*))^{\frac{1}{2}} = \Lambda \tag{3.14}$$

In certain cases, the solution may be obtained by inspection. In any case, Theorem 1 gives upper and lower bounds for Λ

Theorem 1. Suppose $Q(s,t) \geq 0$, s, $t \varepsilon T$.

i) $$\inf_{s\varepsilon T} \sup_{t\varepsilon T} \frac{\sqrt{Q(s,s)}}{Q(s,t)} \geq \Lambda \geq \sup_{t\varepsilon T} \frac{1}{Q(t,t)} \tag{3.15}$$

ii) If there exists a (necessarily unique) s_* for which

$$\sup_{s\epsilon T} \frac{1}{\sqrt{Q(s,s)}} = \frac{1}{\sqrt{Q(s_*,s_*)}} \tag{3.16a}$$

and

$$Q(s_*, s_*) = \inf_{t\epsilon T} Q(s_*, t) \tag{3.16b}$$

then

$$\psi^* = \frac{Q_{s_*}}{||Q_{s_*}||} \tag{3.17}$$

and

$$f^* = \frac{1}{\alpha(s_*)Q(s_*,s_*)} \eta_{s_*} \tag{3.18}$$

Proof: The right hand inequality in i) follows from the Cauchy-Schwartz inequality:

$$\alpha(t) \leq \left\langle \eta_t, f^* \right\rangle_{\mathcal{H}} \leq ||\eta_t||_{\mathcal{H}} ||f^*||_{\mathcal{H}} = \alpha(t)\sqrt{Q(t,t)}\ \Lambda \tag{3.19}$$

The left hand inequality in i) follows by considering the set $S' = \{\frac{Q_s}{||Q_s||_Q}\}, s\epsilon T\} \subset S$. Then

$$\inf_{\psi\epsilon S'} \sup_{t\epsilon T} \frac{1}{\psi(t)} = \inf_{s\epsilon T} \sup_{t\epsilon T} \frac{||Q_s||_Q}{Q_s(t)} = \inf_{s\epsilon T} \sup_{t\epsilon T} \frac{Q(s,s)}{Q(s,t)} \geq \Lambda \tag{3.20}$$

If (3.16) holds, then

$$\frac{1}{Q(s_*,s_*)} = \sqrt{Q(s_*,s_*)} \sup_{t\epsilon T} \frac{1}{Q(s_*,t)} \geq \inf_{s\epsilon t} \sup_{t\epsilon T} \frac{\sqrt{Q(s,s)}}{Q(s,t)} \geq \frac{1}{\sqrt{Q(s_*,s_*)}} \tag{3.21}$$

so that

$$\Lambda = 1/\sqrt{Q(s_*,s_*)} = \sup_t \frac{1}{\psi^*(t)} = \sup_t \frac{||Q_{s_*}||_Q}{Q(s_*,t)} \tag{3.22}$$

and

$$\psi^* = \frac{Q_{s_*}}{||Q_{s_*}||_Q} \tag{3.23}$$

Equation (3.18) follows from (3.17) by the remarks following (3.12) and by the fact that the solution f^* to (3.13) is given by (2.14).

As an example of the application of part ii) of the Theorem, we seek the solution to the problem: Find $f^* \varepsilon W^{2,2}$ to minimize

$$J(f) = \lambda_0 f^2(0) + \lambda_1 (f'(0))^2 + \int_0^1 (w(t)f''(t))^2 dt \tag{3.24}$$

subject to

$$\alpha(t) \le f'(t) \ , \ t\varepsilon T = [\tfrac{1}{2},1] \ , \tag{3.25}$$

where λ_0, $\lambda_1 > 0$, and α and w are given positive functions. The reproducing kernel for $W^{2,2}$ with $||f||^2 = J(f)$ is

$$R(s,t) = \frac{1}{\lambda_0} + \frac{st}{\lambda_1} + \int_0^{\min(s,t)} \frac{(s-u)(t-u)}{w^2(u)} du \tag{3.26}$$

To obtain the representer η_t of the continuous linear functional N_t' defined by $N_t' f \rightarrow f'(t)$, we use the fact that

$$\eta_t(s) = \left\langle \eta_t, R_s \right\rangle_{\mathcal{H}} = N_t' R_s = \frac{\partial}{\partial t} R(s,t) = \frac{s}{\lambda_1} + \int_0^{\min(s,t)} \frac{(s-u)}{w^2(u)} du \ . \tag{3.27}$$

Here

$$Q(s,t) = \frac{1}{\alpha(s)\alpha(t)} \left\langle \eta_s, \eta_t \right\rangle_{\mathcal{H}} = \frac{1}{\alpha(s)\alpha(t)} \frac{\partial^2}{\partial s \partial t} R(s,t)$$

$$= \frac{1}{\alpha(s)\alpha(t)} \left[\frac{1}{\lambda_1} + q(\min(s,t)) \right] , \quad s, t \varepsilon T \tag{3.28}$$

where

$$q(s) = \int_0^s \frac{1}{w^2(u)} du .$$

If $\alpha(t)$ is non-increasing on $T = [\frac{1}{2},1]$ then, with $s_* = \frac{1}{2}$

$$\sup_{t \varepsilon T} \frac{1}{\sqrt{Q(t,t)}} = \frac{1}{\sqrt{Q(s_*,s_*)}}$$

$$Q(s_*,s_*) = \inf_{t \varepsilon T} Q(s_*,t). \tag{3.29}$$

Thus Q with $s_* = \frac{1}{2}$ satisfies (3.16) and

$$f^* = \frac{1}{\alpha(s_*)Q(s_*,s_*)} \eta_{s_*}$$

$$f^*(s) = \frac{\alpha(\frac{1}{2})}{(\frac{1}{\lambda_1} + q(\frac{1}{2}))} \left[\frac{s}{\lambda_1} + \int_0^{\min(s,\frac{1}{2})} \frac{(s-u)}{w^2(u)} du \right] \tag{3.30}$$

Similarly, if

$$\frac{(1 + \lambda_1 q(t))}{(1 + \lambda_1 q(1))} \geq \frac{\alpha(t)}{\alpha(1)} \geq \left(\frac{\alpha(t)}{\alpha(1)}\right)^2 \tag{3.31}$$

then Q with $s^* = 1$ satisfies (3.16) and

$$f^* = \frac{1}{\alpha(1)Q(1,1)} \eta_1 \tag{3.32}$$

$$f^*(s) = \frac{\alpha(1)}{(\frac{1}{\lambda_1} + q(1))} \left\{ \frac{s}{\lambda_1} + \int_0^s \frac{(s-u)}{w^2(u)} du \right\} \quad 0 \leq s \leq 1$$

4. Properties of the approximate solution. Let f_n^* be the (unique) solution to the problem

$$\text{minimize} \quad ||f||_{\mathcal{H}} \tag{4.1a}$$

subject to

$$\alpha(t) \leq \langle f, \eta_t \rangle_{\mathcal{H}} \; , \; t \varepsilon T_n \; , \tag{4.1b}$$

where $T_n = \{s_1, s_2, \ldots, s_n, s_i \varepsilon T\}$.

f_n^* is obtained as follows: For any $f \varepsilon \mathcal{H}$, we may write

$$f = \sum_{i=1}^{n} c_i \eta_{s_i} / \alpha(s_i) + \rho$$

where $\langle \eta_t, \rho \rangle_{\mathcal{H}} = 0$, $t \varepsilon T_n$. Then 4.1a and 4.1b become

$$\text{minimize} \quad cQ_n c' + \langle \rho, \rho \rangle_{\mathcal{H}} \tag{4.2a}$$

subject to

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \leq Q_n c' \tag{4.2b}$$

where $c = (c_1, c_2, \ldots, c_n)$ and Q_n is the $n \times n$ matrix (of full rank) with i jth element

$$[Q_n]_{ij} = \frac{1}{\alpha(s_i)\alpha(s_j)} \left\langle \eta_{s_i}, \eta_{s_j} \right\rangle_{\mathcal{H}} = Q(s_i, s_j)$$

It is obvious that ρ must be 0 in (4.2a). Thus the problem is reduced to finding c to minimize $cQ_n c'$ subject to (4.2b), a standard quadratic programming problem. Let $c^* = (c_1^*, c_2^*, \ldots, c_n^*)$ be the (unique) solution to this problem.

Thus, $f_n^* \varepsilon V$ is given by

$$f_n^* = \sum_{i=1}^{n} c_i^* \eta_{s_i} \tag{4.3}$$

and

$$g_n^* = \sum_{i=1}^{n} c_i^* Q_{s_i} \tag{4.4}$$

is that element in $\mathcal{H}_Q$ corresponding to f_n^* under the isomorphism of (2.7). g_n^* is the solution to the problem

$$\text{minimize} \quad ||g||_Q \tag{4.5a}$$

subject to

$$1 \leq \left\langle g, Q_t \right\rangle = g(t) \ , \quad t \varepsilon T_n \ . \tag{4.5b}$$

Let $f^* \epsilon V$ and $g^* \epsilon \mathcal{H}_Q$ be the unique solutions to the problems of (2.1), (2.2) and (2.11) respectively. Note that $||g_n^*||_Q \leq ||g^*||_Q = \Lambda$.

It is our purpose to study the behavior of $||f_n^* - f^*||_{\mathcal{H}}$. Since $||f_n^* - f^*||_{\mathcal{H}} = ||g_n^* - g^*||_Q$, we may restrict our attention to $||g_n^* - g^*||_Q$.

For notational simplicity we let $T = \{\xi_i\}_{i=1}^{\ell} \cup [a,b]$ where $\{\xi_i\}$ are ℓ isolated points not in $[a,b]$. The argument below may be carried through for $[a,b]$ replaced by any finite union of closed bounded intervals. The behavior of $||g_n^* - g^*||_Q$ proceeds by studying Q. We have the following

Lemma 2. Let $T = D \cup E$, $T_n = D \cup E_n$, where E is a closed, bounded interval $[a,b]$ and $E_n = \{a = t_1 < t_2 < \cdots < t_n = b\}$, with $\Delta = \max|t_{i+1} - t_i|$. Let Q have continuous mixed partial derivatives of all orders to $2p - 2$, and bounded mixed partial derivatives to order $2p - 1$ on $E \times E$.

Then there exists $k = k(Q)$ depending only on Q such that

$$g_n^*(t) \geq 1 - k||g_n^*||_Q \, \Delta^{\min(p-\frac{1}{2},\, 2)} \tag{4.9a}$$

$$t \epsilon T$$

$$\geq 1 - k\Lambda \, \Delta^{\min(p-\frac{1}{2},\, 2)} \tag{4.9b}$$

Proof. Since $D \subset T_n$, it is only necessary to consider $t \epsilon E$. Let $\{d_i\}_{i=1}^n$ be any set of real numbers with $d_i \geq 0$, $\sum_{i=1}^n d_i = 1$. Then since $g_n^*(t_i) \geq 1$, $\sum_{i=1}^n d_i g_n^*(t_i) \geq 1$, and

$$|g_n^*(t) - \sum_{i=1}^{n} d_i g_n^*(t_i)| = |\langle g_n^*, Q_t - \sum_{i=1}^{n} d_i Q_{t_i} \rangle_Q|$$

$$\leq ||g_n^*||_Q \, ||Q_t - \sum_{i=1}^{n} d_i Q_{t_i}||_Q \tag{4.10}$$

Thus

$$g_n^*(t) \geq 1 - ||g_n^*||_Q \, ||Q_t - \sum_{i=1}^{n} d_i Q_{t_i}||_Q \tag{4.11}$$

For $p = 1, 2$, we find, for each t, a set of $\{d_i\}$ for which

$$||Q_t - \sum_{i=1}^{n} d_i Q_{t_i}||_Q \leq k(Q)\Delta^{p-\frac{1}{2}} \tag{4.12}$$

For $t \varepsilon [t_j, t_{j+1}]$, let $d_i = 0$, $i \neq j, j+1$, and

$$d_j = \frac{(t_{j+1} - t)}{(t_{j+1} - t_j)}, \quad d_{j+1} = \frac{(t - t_j)}{(t_{j+1} - t_j)} \tag{4.13}$$

Then, for $t\ [t_j, t_{j+1}]$,

$$||Q_t - d_j Q_{t_j} - d_{j+1} Q_{t_{j+1}}||^2 = Q(t,t) - 2d_j Q(t_j,t) - 2d_{j+1} Q(t_{j+1},t)$$

$$+ d_j^2 Q(t_j,t_j) + 2d_j d_{j+1} Q(t_j,t_{j+1}) + d_{j+1}^2 Q(t_{j+1},t_{j+1}). \tag{4.14}$$

For $p = 1$, Q has a bounded first derivative in each variable and

$$Q(u,v) = Q(t_j, t_j) + (u-t_j)k_1 + (v-t_j)k_2 \tag{4.15}$$

where k_1 and k_2 are bounded in absolute value by

$$\max_{t,x} \left|\frac{\partial}{\partial x} Q(t, x)\right| \tag{4.16}$$

Substituting (4.15) into (4.14) with u, v replaced by t_j, t and t_{j+1} as appropriate, the zero th order terms all drop out, leaving only terms involving $(u-t_1)$ and $(v-t_1)$. So, for $t \varepsilon [t_j, t_{j+1}]$, $\exists k = k(Q)$ such that

$$||Q_t - d_j Q_{t_j} - d_{j+1} Q_{t_{j+1}}||_Q^2 \leq k^2 \Delta \qquad p=1 \tag{4.17}$$

For p = 2, Q has continuous mixed partial derivatives to order 2 with the 3rd order mixed partial derivatives bounded. Thus we may write

$$\begin{aligned} Q(u, v) = Q(t_j, t_j) &+ [(u-t_j) + (v-t_j)]\, \alpha_1 \\ &+ \left[\frac{(u-t_j)^2}{2!} + \frac{(v-t_j)^2}{2!}\right] \alpha_2 \\ &+ \frac{(u-t_j)(v-t_j)}{2!}\, \beta \\ &+ \sum_{i=0}^{3} \frac{(u-t_j)^i}{i!} \frac{(v-t_j)^{3-i}}{(3-i)!}\; k_{i+3} \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} \alpha_1 &= \frac{\partial}{\partial x} Q(t_j, x)|_{x=t_j} \\ \alpha_2 &= \frac{\partial^2}{\partial x^2} Q(t_j, x)|_{x=t_j} \\ \beta &= \frac{\partial^2}{\partial x \partial y} Q(x, y)|_{x=y=t_j} \end{aligned} \tag{4.19}$$

and the k_i, $i = 3, 4, 5, 6$ are all bounded in absolute value by the bound on the absolute third mixed partial derivatives of Q.

Substituting (4.18) into (4.14), some tedious calculations, partly reproduced in the Appendix, show that all but the third order terms cancel out, giving the existence of a $k = k(Q)$ such that

$$||Q_t - d_j Q_{t_j} - d_{j+1} Q_{t_{j+1}}||_Q^2 \leq k^2 \Delta^3 \tag{4.20}$$

If $g \in \mathcal{H}_Q$ with $p \geq 3$, then, by the existence and continuity of the mixed partial derivatives of Q to order four, we have

$$\begin{aligned} |\sup_{t\in E} g''(t)| &\leq \sup_{t\in E} |\lim_{\delta\to 0} (g(t+\delta) - 2g(t) + g(t-\delta))/2\delta| \\ &\leq \sup_t |\lim_{\delta\to 0} \langle g, (Q_{t+\delta} - 2Q_t + Q_{t-\delta})/2\delta \rangle_Q| \\ &\leq ||g||_Q \sup_t |\lim_{\delta\to 0}| \, ||(Q_{t+\delta} - 2Q_t + Q_{t-\delta})/2\delta||_Q \\ &= ||g||_Q \sup_t \left| \frac{\partial^4}{\partial^2 u \partial^2 v} Q(u, v)|_{u=v=t} \right|^{\frac{1}{2}} \end{aligned} \tag{4.21}$$

We may now expand g in a Taylor series as

$$g(t_i) = g(t) + (t_i - t)g'(t) + \int_t^{t_i} (t_i - u)g''(u)du \, . \tag{4.22}$$

For $t\varepsilon[t_j, t_{j+1}]$, choose $\{d_i\}$ as in (4.13). Then

$$|g(t) - d_j g(t_j) - d_{j+1} g(t_{j+1})| = |d_j \int_t^{t_j} (t_j-u)g''(u)du + d_{j+1} \int_t^{t_{j+1}} (t_{j+1}-u)g''(u)du$$

$$\leq k||g||_Q \frac{(t_{j+1}-t_j)^2}{2} \leq \frac{k}{2}||g||_Q \Delta^2 \qquad (4.23a)$$

where

$$k = \sup_t \left[\frac{\partial^4}{\partial^2 u \partial^2 v} Q(u,v)|_{u=v=t} \right] \qquad (4.23b)$$

Thus we have proved (4.9a) for $p \geq 3$.

We may ask if these rates may be improved upon. Suppose

(i) $\frac{\partial^\ell}{\partial t^\ell} Q(t,t')$ exists and is continuous on $E\times E$ for $t \neq t'$, $\ell = 0, 1, 2, \ldots, 2p$, $\frac{\partial^\ell}{\partial t^\ell} Q(t,t')$ exists and is continuous on $E\times E$ for $\ell = 0, 1, 2, \ldots, 2p - 2$,

(ii) $\lim_{t\uparrow t'} \frac{\partial^{2p-1}}{\partial t^{2p-1}} Q(t,t')$ and $\lim_{t\downarrow t'} \frac{\partial^{2p-1}}{\partial t^{2p-1}} Q(t,t')$ exist and are bounded for all $t'\varepsilon E$.

It may be shown (see [7] and [8]) that if (i) and (ii) hold, that there exist constants, $\{e_i\}$ for which

$$||Q_t - \sum_{i=1}^n e_i Q_{t_i}||_Q = O(\Delta^{p-\frac{1}{2}}) , \qquad (4.24)$$

but, evidently, this rate cannot be improved upon.

We now examine the case for $p = 3$ to see if some improvement there is possible there. Consider $\mathcal{H}_Q = W^{3,2}$ with the norm given by

$$||g||_Q^2 = \sum_{j=0}^{2} (g^j(0))^2/j! + \int_0^1 (g^{iii}(u))^3 du \qquad (4.25)$$

Here we may expand g in a Taylor series with remainder to order 2 and have

$$|g(t) - \sum_{i=1}^{n} g(t_i)d_i(t)| = |g(t) - \sum_{i=1}^{n} \{(g(t) + (t_i-t)g'(t) + (t_i-t)^2/2!g''(t) + \int_t^{t_i} \frac{(t_i-u)^2}{2!} g^{iii}(u)du\, d_i(t)\}| \tag{4.26}$$

In order that the right hand side is $o(\Delta^2)$ we need, in addition to $\sum_{i=1}^{n} d_i(t) = 1$, $d_i(t) \geq 0$, that

$$0 = \sum_{i=1}^{n} (t_i-t)d_i(t) = \sum_{i=1}^{n} (t_i-t)^2 d_i(t)$$

or

$$t = \sum_{i=1}^{n} t_i d_i(t)$$

$$t^2 = \sum_{i=1}^{n} t_i^2 d_i(t)$$

But if $\{d_i(t)\}_{i=1}^{n}$ is viewed (for fixed t), as a probability distribution on the points $t_1, t_2, \ldots, t_n$, then t and t^2 are its first and second moments. But then, the only way that $\{d_i(t)\}_{i=1}^{n}$ can be a probability distribution is for the variance to be zero, that is, $d_j(t) = 1$ for some j and 0 otherwise, and then we must have $t_j = t$. Thus, there exists a set of $\{d_i\}$ to achieve $o(\Delta)$ only if $t_j = t$ for some j. We are now able to prove the following

Theorem 2. Let T, T_n and Q be as in Lemma 2. Then

$$||f_n^* - f^*|| = O(\Delta^{\frac{1}{2}\min(p-\frac{1}{2},\, 2)}) \tag{4.27}$$

Proof: We may replace $||f_n^* - f^*||_{\mathcal{H}}$ by $||g_n^* - g^*||_Q$. Let k be as in (4.9a) and let $\gamma_n = k\Lambda\Delta^{\min(p-\frac{1}{2}, 2)}$. Then by Lemma 2 we have

$$\inf_t g_n^*(t) \geq 1 - \gamma_n \tag{4.28}$$

Since

$$\frac{g_n^*(t)}{1-\gamma_n} \geq 1 \quad \text{and} \quad ||g_n^*||_Q \leq ||g^*||_Q \tag{4.29}$$

we must have

$$||\frac{g_n^*}{(1-\gamma_n)}||_Q = \frac{1}{(1-\gamma_n)} ||g_n^*||_Q \geq \Lambda \geq ||g_n^*||_Q \geq \Lambda(1-\gamma_n) \tag{4.30}$$

and so

$$||g_n^*||_Q = \Lambda(1 - \theta\gamma_n) \tag{4.31}$$

for some $\theta\varepsilon[0,1]$. Hence, letting $g_n^* = ||g_n^*||_Q \psi_n^*$ and $g^* = \Lambda\psi^*$ gives

$$||g_n^* - g^*||_Q^2 = ||\Lambda(1 - \theta\gamma_n)\psi_n^* - \Lambda\psi^*||_Q^2$$

$$= \Lambda^2\{(1 - \theta\gamma_n)^2 - 2(1 - \theta\gamma_n)\left\langle\psi_n^*, \psi^*\right\rangle_Q + 1\} . \tag{4.32}$$

Now

$$\inf_t \psi_n^*(t) = \inf_t \frac{g_n^*(t)}{||g_n^*||_Q} \geq \frac{1-\gamma_n}{\Lambda} = \frac{1}{\Lambda+\varepsilon_n} \tag{4.33}$$

where

$$\varepsilon_n = \frac{\Lambda\gamma_n}{(1-\gamma_n)} . \tag{4.34}$$

Therefore, Lemma 1 gives

$$\left\langle \psi_n^*, \psi^* \right\rangle_Q \geq 1 - 4\gamma_n/(1-\gamma_n) \qquad (4.35)$$

and (4.32) then gives

$$||g_n^* - g^*||_Q^2 \leq \Lambda^2\{(1-\theta\gamma_n)^2 - 2(1-\theta\gamma_n)(1 - 4\gamma_n/(1-\gamma_n)) + 1\}$$

$$= 8\Lambda^2(\gamma_n + o(\gamma_n))$$

$$= 8k\Lambda^3(\Delta^{\min(p-\frac{1}{2},2)}(1 + o(1))) \qquad (4.36)$$

or

$$||g_n^* - g^*||_Q = O(\Delta^{\frac{1}{2}\min(p-\frac{1}{2},\ 2)})$$

Appendix

Table For Calculations, Substitution of (4.18) into (4.14).

$$\delta_1 = (t - t_j),\ \delta_2 = (t_{j+1} - t),\ \delta = \delta_1 + \delta_2 = (t_{j+1} - t_j)$$

				Coefficients in (4.17)			
				$Q(t_j,t_j)$	α_1	$\frac{1}{2!}\alpha_2$	$\frac{1}{2!}\beta$
(u,v)	Coefficient of $Q(u,v)$ in (4.14)	$(u-t_j)$	$(v-t_j)$		$(u-t_j)+(v-t_j)$	$(u-t_j)^2+(v-t_j)^2$	$(u-t_j)(v-t_j)$
(t,t)	$+1$	δ_1	δ_1	1	$2\delta_1$	$2\delta_1^2$	δ_1^2
(t_j,t)	$-2\frac{\delta_2}{\delta}$	0	δ_1	1	δ_1	δ_1^2	0
(t_{j+1},t)	$-2\frac{\delta_1}{\delta}$	δ	δ_1	1	$\delta+\delta_1$	$\delta^2+\delta_1^2$	$\delta\delta_1$
(t_j,t_j)	$+\frac{\delta_2^2}{\delta^2}$	0	0	1	0	0	0
(t_j,t_{j+1})	$+2\frac{\delta_1\delta_2}{\delta^2}$	0	δ	1	δ	δ^2	0
(t_{j+1},t_{j+1})	$+\frac{\delta_1^2}{\delta^2}$	δ ,	δ	1	2δ	$2\delta^2$	δ^2

To obtain the coefficient of $Q(t_j,t_j)$ in (4.14) with (4.18) substituted in, multiply entries in column 2 with the corresponding entries under the column headed by $Q(t_j,t_j)$, and add. To obtain the coefficient of α_1, multiply entries in column 2 with the corresponding entries under the column headed by α_1, and add, and similarly for α_2 and β. The results are all 0.

References

[1] David Blackwell and M. A. Girshick, Theory of Games and Statistical Decisions, Wiley, New York, 1954.

[2] James W. Daniel, Convergence of a discretization for constrained spline function problems, SIAM J. Control, 9 (1971) pp 83-96.

[3] George Kimeldorf and Grace Wahba, Some results on Tchebycheffian spline functions, J. Math. Anal. and Applic., 33 (1971), pp 82-95.

[4] P. J. Laurent, Construction of spline functions in a convex set, in Approximations With Special Emphasis on Spline Functions, I. J. Schoenberg, ed., Academic Press, London, 1969, pp 415-446.

[5] O. L. Mangasarian and L. L. Schumaker, Splines via optimal control, in Approximations With Special Emphasis on Spline Functions, J. J. Schoenberg, ed., Academic Press, London, 1969, pp 119-156.

[6] Klaus Ritter, Generalized spline interpolation and non-linear programming, in Approximations With Special Emphasis on Spline Functions, I. J. Schoenberg, ed., Academic Press, London, 1969, pp 75-118.

[7] Grace Wahba, Convergence rates for certain approximate solutions to Fredholm integral equations of the First Kind, to appear, J. Approx. Theory.

[8] Grace Wahba, A class of approximate solutions to linear operator equations, University of Wisconsin, Department of Statistics Technical Report 270, July 1971.