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INTERPOLATING SPLINE METHODS FOR DENSITY ESTIMATION I. EQUI-SPACED KNOTS

by

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2. EXPLICIT EXPRESSIONS FOR THE HISTOSPLINE ESTIMATE, AND OUTLINE OF PROOF OF THE MAIN THEOREM.

Our development of an explicit formula for an interpolating spline will be slightly unorthodox, for the purpose of easing the proofs of the main theorem. The reader may consult [2], [6], [7], [15] and the bibliography [14] for additional background on splines.

We endow $W_2^{(2)}$ with the inner product

$$< F,G > = F(0)G(0) + F'(0)G'(0) + \int_0^1 F''(x)G''(x)dx . \qquad (2.1)$$

$W_2^{(2)}$ is then a reproducing kernel Hilbert space (RKHS) with the reproducing kernel

$$\begin{aligned} Q(s,t) &= 1+st + \int_0^{\min(s,t)} (s-u)(t-u)du \\ &= 1+st + \left(\frac{ts^2}{2} - \frac{s^3}{6}\right), \quad s < t \\ &= 1+st + \left(\frac{st^2}{2} - \frac{t^3}{6}\right), \quad s > t . \end{aligned} \qquad (2.2)$$

Denote this Hilbert space $\mathcal{H}_Q$, with norm $||\cdot||_Q$. The true c.d.f. F is always assumed to be in $\mathcal{H}_Q$, that is, $f = F' \varepsilon W_2^{(1)}$.

Let Q_t be the function defined on [0,1] by

$$Q_t(s) = Q(s,t), \quad s,t \varepsilon [0,1], \qquad (2.3)$$

and let Q_t' be the function defined on [0,1] by

$$Q_t'(s) = \frac{d}{du} Q(s,u)\Big|_{u=t} , \quad s,t \ [0,1] .$$

$$= s + \frac{s^2}{2} , \quad s \leq t$$

$$= s + st - \frac{t^2}{2} , \quad s \geq t . \qquad (2.4)$$

By the properties of RKHS (see, e.g. [10]), $Q_t, Q_t{}' \varepsilon \mathcal{H}_Q$ for each t, and

$$< G,Q_t > = G(t) , \quad G\varepsilon\mathcal{H}_Q, \ t\varepsilon[0,1] \qquad (2.5)$$

$$< G,Q_t' > = G'(t) .$$

Denote the norm in $\mathcal{H}_Q$ by $||\cdot||_Q$. Consider the solution to the problem: Find $G\varepsilon\mathcal{H}_Q$ to min $||G||_Q$ subject to

$$G'(0) = < G,Q_0' > = a_1$$

$$G(0) = < G,Q_0 > = a_0$$

$$G(s_i) = < G,Q_{s_i} > = y_i, \quad i = 1, 2, \ldots, \ell \qquad (2.6)$$

$$G(1) = < G,Q_1 > = b_0$$

$$G'(1) = < G,Q_1' > = b_1$$

Denoting $\bar{s} = (s_1, s_2, \ldots, s_\ell)$, $\bar{y} = (y_1, y_2, \ldots, y_\ell)$, $\bar{a} = (a_1, a_0)$,

$\bar{b} = (b_0, b_1)$, let $S(x) = S(x; \bar{s}; \bar{a}, \bar{y}, \bar{b})$ be the solution to this problem. Then, by observing that $S \varepsilon \mathcal{S}_\ell(\bar{s})$ defined by

$$\mathcal{S}_\ell(\bar{s}) = \text{span}\{Q_0', Q_{s_i}, i = 0, 1, \ldots, \ell+1, Q_1'\} \qquad (2.7)$$

it may be established that

$$S(x; \bar{s}; \bar{a}, \bar{y}, \bar{b}) = (Q_0'(x), Q_0(x), Q_{s_1}(x), \ldots, Q_{s_\ell}(x), Q_1(x), Q_1'(x))Q_{\ell+4}(\bar{a}; \bar{y}; \bar{b})' \qquad (2.8)$$

where $Q_{\ell+4}$ is the $(\ell+4) \times (\ell+4)$ Grammian matrix of the basis for $\mathcal{S}_\ell(\bar{s})$. $Q_{\ell+4}$ is of full rank (see for example [19]) and the entries may be found from (2.5). By observing the nature of the inner product in $\mathcal{H}_Q$, it is easily seen that $S(x; \bar{s}; \bar{a}, \bar{y}, \bar{b})$ is also the solution to: Find $G \varepsilon \mathcal{H}_Q$ to

$$\min \int_0^1 (G''(x))^2 \, dx$$

subject to (2.6). The solution to this problem is well known [15] to be the unique cubic spline satisfying (2.6). It may easily be checked from (2.3), (2.4) and (2.8) that S has the characteristic properties of a cubic spline, viz. S is a polynomial of degree less than or equal to three in each interval $[s_i, s_{i+1}]$, $i = 0, 1, \ldots, \ell$, and S, S' and S'' are continuous.

The density estimate $\hat{f}_n(x)$ that we study is thus given by

$$\hat{f}_n(x) = \frac{d}{dx}\hat{F}_n(x),$$

$$\hat{F}_n(x) = S(x;\ \bar{s}_h;\ \hat{a},\ \bar{F}_n,\ \hat{b}) \qquad (2.9)$$

with

$$\bar{s}_h = (h,\ 2h,\ \ldots,\ \ell h), \qquad (\ell+1)h = 1$$

$$\hat{a} = (\hat{a}_1,\ 0)$$

$$\bar{F}_n = (F_n(h),\ F_n(2h),\ \ldots,\ F_n(\ell h))$$

$$\hat{b} = (1,\ \hat{b}_1).$$

Equation (2.8) is not the computationally best method for computing $\hat{F}_n$, because $Q_{\ell+4}$ is ill-conditioned for large ℓ, however, computing routines for $S(x)$ and $S'(x)$ are commonly available, see, for example [1]. The estimates $\hat{a}_1$ and $\hat{b}_1$ depend on m, (= 1, 2, or 3) and are defined as follows: Let $\ell_{0,\nu}(x)$ be the polynomial of degree m satisfying

$$\ell_{0,\nu}(x) = 1,\ x = \nu h,$$

$$= 0,\ x = jh,\ j \neq \nu,\ j = 0,\ 1,\ \ldots,\ m. \qquad (2.10)$$

and let $\ell_{1,\nu}(x)$ be the polynomial of degree m satisfying

$$\ell_{1,\nu}(x) = 1, \; x = 1-\nu h$$

$$= 0, \; x = 1-jh, \; j \neq \nu, \; j = 0, 1, \ldots, m. \qquad (2.11)$$

Let

$$\hat{a}_1 = \hat{F}_n'(0) = \hat{f}_n(0) = \frac{d}{dx} \sum_{\nu=0}^{m} \ell_{0,\nu}(x)\Big|_{x=0} F_n(\nu h) \qquad (2.12)$$

$$\hat{b}_1 = \hat{F}_n'(1) = \hat{f}_n(1) = \frac{d}{dx} \sum_{\nu=0}^{m} \ell_{1,\nu}(x)\Big|_{x=1} F_n(1-\nu h). \qquad (2.13)$$

$\hat{a}_1$ is the derivative at 0 of the mth degree polynomial interpolating the sample c.d.f. at $0, h, \ldots, mh$, and similarly for $\hat{b}_1$. It follows from (2.8) that $S(x; \bar{s}; \bar{a}, \bar{y}, \bar{b})$ is linear in the entries of $\bar{a}$, $\bar{y}$, and $\bar{b}$, that is

$$S(x; \bar{s}; \bar{a}+\bar{\varepsilon}_a, \bar{y}+\bar{\varepsilon}, \bar{b}+\bar{\varepsilon}_b) = S(x; \bar{s}; \bar{a}, \bar{y}, \bar{b}) + S(x; \bar{s}; \bar{\varepsilon}_a, \bar{\varepsilon}, \bar{\varepsilon}_b)$$

where $\bar{\varepsilon}_a$, $\bar{\varepsilon}$ and $\bar{\varepsilon}_b$ are 2-, ℓ- and 2-vectors, respectively. $\frac{d}{dx} S'(x; \bar{s}, \bar{a}, \bar{y}, \bar{b})$ also has this linearity property.

Let F be the true c.d.f., and let $\tilde{F}$ be the cubic spline of interpolation to F, with knots jh, $j = 1, 2, \ldots, \ell$, and matching F and F' at the boundaries, that is

$$\tilde{F}(x) = S(x; \bar{s}_h; \bar{F}_a, \bar{F}_h, \bar{F}_b)$$

where

$$\bar{F}_a = (F'(0), 0)$$

$$\bar{F}_h = (F(h), F(2h), \ldots, F(\ell h))$$

$$\bar{F}_b = (1, F'(1)).$$

Then we may write

$$f(x) - \hat{f}_n(x) = \frac{d}{dx}(F(x) - \hat{F}_n(x))$$

$$= \frac{d}{dx}(F(x) - \tilde{F}(x)) + \frac{d}{dx}(\tilde{F}(x) - \hat{F}_n(x)) \quad (2.14)$$

$$= \frac{d}{dx}(F(x) - \tilde{F}(x)) + \frac{d}{dx} H_n(x), \quad (2.15)$$

where

$$H_n(x) = S(x; \bar{s}_h; \bar{\varepsilon}_a, \bar{\varepsilon}, \bar{\varepsilon}_b) \quad (2.16)$$

and

$$\bar{\varepsilon}_a = (\varepsilon_0', 0) \quad , \quad \varepsilon_0' = F'(0) - \hat{a}_1,$$

$$\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell), \quad \varepsilon_j = F(jh) - F_n(jh),\ j = 1, 2, \ldots, \ell, \quad (2.17)$$

$$\bar{\varepsilon}_b = (0, \varepsilon_{\ell+1}') \quad , \quad \varepsilon_{\ell+1}' = F'(1) - \hat{b}_1 .$$

The first term on the right of (2.16) which we shall call the bias term, is non-random and depends only on how well F can be approximated by an interpolating cubic spline. The second, or variance term is a (linear) function of the random variables ε_0', $\varepsilon_{\ell+1}'$ and ε_i, $i = 1, 2, \ldots, \ell$.

Then, as usual,

$$E(f(x) - \hat{f}_n(x))^2 \leq 2\left(\frac{d}{dx}(F(x) - \tilde{F}(x))\right)^2 + 2\,E\left(\frac{d}{dx}H_n(x)\right)^2 . \qquad (2.18)$$

Bounds on the absolute bias, $|\frac{d}{dx}(F(x) - \tilde{F}(x))|$ appear in the approximation theory literature in various forms, for equally spaced, as well as arbitrarily spaced knots. If $F^{(m+1)} \varepsilon \mathcal{L}_2[0,1]$, then it is known ([18], Theorems 5.1, 5.2 and 5.3) that, for $m = 1, 2, 3$,

$$\sup_x |\frac{d}{dx}(F(x) - \tilde{F}(x))| \leq K_2(m)\, ||F^{(m+1)}||_2\, h^{m-1/2} \qquad (2.19)$$

where $||\cdot||_p$ is the $\mathcal{L}_p$ norm, and $K_2(m)$ is a constant depending on m. Generalizations of (2.19) to arbitrary m are given when $\tilde{F}$ is replaced by an interpolating spline of higher degree. For $F^{(m+1)} \varepsilon \mathcal{L}_\infty[0,1]$, $m = 3$, [9] gives

$$\sup_x |\frac{d}{dx}(F(x) - \tilde{F}(x)| \leq K_\infty(m)\, ||F^{(m+1)}||_\infty\, h^m, \qquad (2.20)$$

and [9] is easily extendable to $m = 1, 2$. (For earlier results, see [13]). Some information about generalizations of (2.20) up to, but not beyond $m = 5$ are known [4]. We would like to have the result

$$F^{(m+1)} \varepsilon \mathscr{L}_p \Rightarrow \sup_x \left| \frac{d}{dx} (F(x) - \tilde{F}(x)) \right| \leq K_p(m) \; ||F^{(m+1)}||_p \; h^{m-1/p}, \quad p \geq 1, \tag{2.21}$$

or, equivalently, $f \varepsilon W_p^{(m)} \Rightarrow$

$$\sup_x \left(\frac{d}{dx} (F(x) - \tilde{F}(x)) \right)^2 \leq \frac{A}{2} \; ||f^{(m)}||_p^2 \; h^{2m-2/p}, \quad p \geq 1 \tag{2.22}$$

where $A = A(m,p)$. We are not aware of such results for $p \neq 2$ or ∞. We provide a proof of (2.22) good for $m = 2, 3, 1 \leq p \leq 2$. In the proof, the dependency on the knots $\{s_i\}_{i=1}^{\ell}$ is retained so that the results may be used in a sequel paper where the knots are determined by the order statistics. Combining these results will give us a bound on the bias for

$m = 1, \quad p = 2 \quad , \infty$

$m = 2, \quad 1 \leq p \leq 2, \infty$

$m = 3, \quad 1 \leq p \leq 2, \infty$.

The establishment of bounds on the bias term is tedious for cubic splines, and we are unable to do it for higher degree splines. It will be shown that, for x not in a neighborhood of 0 or 1

$$E\left(\frac{d}{dx} H_n(x)\right)^2 \leq \frac{B}{2} \frac{1}{nh} \tag{2.23}$$

where B is a constant to be given. Then we will have

$$E(f(x) - \tilde{f}_n(x))^2 \le A \, ||f^{(m)}||_p^2 \, h^{2m-2/p} + B \frac{1}{nh} . \qquad (2.24)$$

The right hand side of (2.24) is minimized (see [12]) by taking $h = k_n/n$ with

$$k_n = \left[\frac{1}{(2m-2/p)} \frac{B}{||f^{(m)}||_p^2 \, A} \right]^{1/(2m+1 - 2/p)} \cdot n^{(2m-2/p)/(2m+1 - 2/p)} . \qquad (2.25)$$

Then, we will have the main result, which is:

$$E[f(x) - \hat{f}_n(x)]^2 \le D \, n^{-(2m-2/p)/(2m+1 - 2/p)} \qquad (2.26)$$

where

$$D = \frac{(2m+1 - 2/p)}{(2m-2/p)^{(2m-2/p)}} \left(||f^{(m)}||_p^2 \, AB^{2m-2/p} \right)^{1/(2m+1 - 2/p)} . \qquad (2.27)$$

Details of these assertions are in the next section.

3. PROOF OF THE MAIN THEOREM

3.1 Bounds on the Bias Term

The case $m = 3$, $p = \infty$ is covered by

Propostion 1. Let $F^{(iv)} \varepsilon \mathscr{L}_\infty[0,1]$. Then

$$\left(\frac{d}{dx}(F(x) - \tilde{F}(x))\right)^2 \le \left(\frac{3}{16}\right)^2 \|F^{(iv)}\|_\infty h^6 . \tag{3.1}$$

Proof: This is Theorem 2 of [9], the proof there may be extended from $F^{(iv)}$ continuous to $F^{(iv)} \varepsilon \mathscr{L}_\infty$.

The case $m = 1$ or 2 and $p = \infty$ is covered by

Proposition 2. Let $F^{(iii)} \varepsilon \mathscr{L}_\infty[0,1]$. Then

$$\left(\frac{d}{dx}(F(x) - \tilde{F}(x)\right)^2 \le \left(\frac{9}{4}\right)^2 \|F^{(iii)}\|_\infty^2 h^4 . \tag{3.2}$$

Suppose only that $F^{(ii)} \varepsilon \mathscr{L}_\infty[0,1]$. Then

$$\left(\frac{d}{dx}(F(x) - \tilde{F}(x)\right)^2 \le \left(\frac{9}{2}\right)^2 \|F^{(ii)}\|^2 h^2 .$$

Proof: This may be proved from the argument in [9] by following the proof of Theorem 2 in [9], and noting that, if $F^{(iii)} \varepsilon \mathscr{L}_\infty$, then r_i of [9], equation (8) is bounded by $3h|\sup_\xi F^{(iii)}(\xi)|$, and if $F^{(ii)} \varepsilon \mathscr{L}_\infty$, then r_i of [9], equation (8) is bounded by $6|\sup_\xi F^{(ii)}(\xi)|$.

The next series of Lemmas result in a Theorem which provides bounds on the bias for $m = 1$, $p = 2$, and $m = 2, 3$, $1 \le p \le 2$.

Lemma 1.

$$\left(\frac{d}{dx}\left(F(x) - \tilde{F}(x)\right)\right)^2 \leq ||Q'_x - \tilde{Q}'_x||_Q^2 \; ||F - \tilde{F}||_Q^2 \qquad (3.4)$$

where $\tilde{Q}'_x$ is the projection of Q'_x in $\mathcal{H}_Q$ onto $\mathcal{S}_\ell(\bar{s})$.

Proof:

$$\left|\frac{d}{dx}\left(F(x) - \tilde{F}(x)\right)\right| = \left|< Q'_x, F-\tilde{F} >\right| = \left|< Q'_x - \tilde{Q}'_x, F-\tilde{F} >\right| . \qquad (3.5)$$

Lemma 2.

$$||Q'_x - \tilde{Q}'_x||_Q^2 \leq \frac{1}{3} h \qquad (3.6)$$

Proof: See Appendix.

Lemma 3.

Let $F^{(iv)} \epsilon \mathcal{L}_p[0,1]$, $1 \leq p \leq 2$. Then

$$||F-\tilde{F}||_Q^2 \leq ||F^{(iv)}||_p^2 \; h^{5-2/p} \qquad (3.7)$$

Proof: See Appendix.

Lemma 4.

Let $F^{(iii)} \epsilon \mathcal{L}_p[0,1]$, $1 \leq p \leq 2$. Then

$$||F-\tilde{F}||_Q^2 \leq (1/3)\ ||F^{(iii)}||_p^2\ h^{3-2/p} \tag{3.8}$$

Proof: See Appendix.

Theorem 1.

Let $f^{(m)} \varepsilon \mathcal{L}_p$ for $m = 1,\ p = 2$, and $m = 2, 3,\ 1 \leq p \leq 2$. Then

$$\left(\frac{d}{dx}\ (F(x) - \tilde{F}(x))\right)^2 \leq A\ ||f^{(m)}||_p^2\ h^{2m-2/p} \tag{3.9}$$

where

$$A = \frac{1}{3}, \quad m = 1, \quad p = 2$$

$$A = \frac{1}{9}, \quad m = 2,\ 1 \leq p \leq 2$$

$$A = \frac{1}{3}, \quad m = 3,\ 1 \leq p \leq 2\ .$$

Proof: The result follows upon combining Lemmas 1, 2, 3 and 4.

3.2. Bounds on the Variance Term

We seek a bound on

$$E[\frac{d}{dx}\ H_n(x)]^2$$

where

$$H_n(x) = S(x; \bar{s}_h, \bar{\varepsilon}_a, \bar{\varepsilon}, \bar{\varepsilon}_b) \tag{3.10}$$

and

$$\bar{s}_h = (h, 2h, \ldots, \ell h) ,$$

$$\bar{\varepsilon}_a = (\varepsilon_0', \varepsilon_0), \quad , \quad \varepsilon_0 = 0, \ \varepsilon_0' = F'(0) - \hat{a}_1,$$

$$\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell), \quad \varepsilon_j = \tilde{F}(jh) - \hat{F}_n(jh) = F(jh) - \hat{F}_n(jh), \ j = 1, 2, \ldots, \ell,$$

$$\varepsilon_b = (\varepsilon_{\ell+1}, \varepsilon_{\ell+1}') \quad , \ \varepsilon_{\ell+1} = 0, \ \varepsilon_{\ell+1}' = F'(1) - \hat{b}_1 .$$

Lemma 5 bounds the derivative of a cubic spline in terms of h and the data $\bar{\varepsilon}_a, \bar{\varepsilon}, \bar{\varepsilon}_b$.

<u>Lemma 5.</u>

For $jh \leq x < (j+1)h, \ j = 0, 1, \ldots, \ell,$

$$\left|\frac{d}{dx} H_n(x)\right| \leq 8 \left\{ \sum_{i=0}^{\ell} c_i \frac{|\psi_i|}{h} + \frac{1}{2^{j+1}} \left| \varepsilon_0' \right| + \frac{1}{2^{\ell+2-j}} \left| \varepsilon_{\ell+1}' \right| \right\} \tag{3.11}$$

where

$$\psi_i = \varepsilon_{i+1} - \varepsilon_i = [F((j+1)h) - F(jh)] - [F_n((j+1)h) - F_n(jh)]$$

$$c_i = \frac{1}{2^{|i-j|+1}} + \frac{1}{2^{|i+1-j|+1}}, \quad i = 0, 1, \ldots, \ell, \quad i \neq j$$

$$c = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{8} .$$

<u>Proof:</u> See Appendix.

Bounds on $E[\frac{d}{dx} H_n(x)]^2$ may now be found by bounding the random variables on the right of (3.11).

Now

$$\psi_j = \varepsilon_{j+1} - \varepsilon_j = [F((j+1)h) - F(jh)] - [\hat{F}_n((j+1)h) - \hat{F}_n(jh)]$$

$$= p_j - \frac{\text{\# of observations between } jh \text{ and } (j+1)h}{n}$$

$$p_j = \int_{jh}^{(j+1)h} f(\xi)d\xi < h\Lambda$$

where

$$\Lambda = \sup_{\xi} f(\xi) .$$

Thus $n(p_j - \psi_j)$ is binomial $B(n,p_j)$ and so

$$E(\psi_j)^2 = \frac{1}{n} p_j(1-p_j) \leq \frac{\Lambda h}{n} . \qquad (3.12)$$

To complete the bound on the variance term, we need to know $E(\varepsilon_0')^2 \equiv E(F'(0) - \hat{a}_1)^2$ and $E(\varepsilon_{\ell+1}')^2 \equiv E(F'(1) - \hat{b}_1)^2$. The answer is given by

Lemma 6. Let $\hat{a}_1$ and $\hat{b}_1$ be given by (2.12) and (2.13) for $m = 1, 2, 3$. Then

$$\left.\begin{array}{l} E(F'(0) - \hat{a}_1)^2 \\ \\ E(F'(1) - \hat{b}_1)^2 \end{array}\right\} \leq \alpha||f^{(m)}||_p^2 \, h^{2m-2/p} + \beta \frac{\Lambda}{nh} \qquad (3.13)$$

where

$$\begin{aligned} \alpha &= 8m^{2m-2/p} / [(m-1)!]^2 \\ \beta &= 2m^3(m+1)^2\Lambda . \end{aligned} \qquad (3.14)$$

Proof: See Appendix.

Note that, if $x = 0$, or $x = 1$,

$$E(f(0) - \hat{f}_n(0))^2 = E(F'(0) - \hat{a}_1)^2$$

$$E(f(1) - \hat{f}_n(1))^2 = E(F'(1) - \hat{b}_1)^2$$

and the mean square error is given by the right hand side of (3.13).

For $x \neq 0, 1$, we combine Lemmas 5, 6 and (3.12) to obtain

$$E\left(\frac{d}{dx} H_n(x)\right)^2 \leq 8^2 \left\{ 2 \sum_{r,s=0}^{\ell} c_r c_s \, E \frac{|\psi_r \psi_s|}{h^2} + 2\left(\frac{1}{2^{j+1}} + \frac{1}{2^{\ell+2-j}}\right)\left(\alpha \, ||f^{(m)}||_p^2 \, h^{2m-2/p} + \beta \frac{1}{nh}\right) \right\}$$

$$\leq 8^2 \left\{ 2(3\tfrac{1}{8})^2 \frac{\Lambda}{nh} + 2\left(\frac{1}{2^{(x/h)}} + \frac{1}{2^{(1-x)/h}}\right) \left(\alpha||f^{(m)}||_p^2 \, h^{2m-2/p} + \beta \frac{1}{nh}\right) \right\} \tag{3.15}$$

Note that if x is bounded away from 0 and 1, then $\left[\frac{1}{2^{(x/h)}} + \frac{1}{2^{(1-x)/h}}\right] \to 0$ rapidly as $h \to 0$.

3.3. Final Result

Summarizing the results from (3.1), (3.2), (3.3), (3.9) and (3.15) gives

$$E(f(x) - \hat{f}_n(x))^2 \leq A||f^{(m)}||_p^2 \, h^{2m-2/p} + B \frac{1}{n}h$$

where

$$A = 2\left[A' + \alpha\left(\frac{2}{2^{x/h}} + \frac{2}{2^{(1-x)/h}}\right)\right]$$

$$B = 2\left[B' + 64\beta\left(\frac{2}{2^{x/h}} + \frac{2}{2^{(1-x)/h}}\right)\right] \tag{3.16}$$

and

$$A' = (9/2)^2 \quad m = 1, \quad p = \infty$$

$$(9/4)^2 \quad m = 2, \quad p = \infty$$

$$(3/16)^2 \quad m = 3, \quad p = \infty$$

$$1/3 \quad m = 1, \quad p = 2$$

$$1/9 \quad m = 2, \ 1 \leq p \leq 2$$

$$4/3 \quad m = 3, \ 1 \leq p \leq 2$$

$$B' = 2 \cdot (8 \cdot 3^{\frac{1}{8}})^2$$

and α and β are given by (3.14). We have thus proved the following.

Theorem 2.

Suppose f has its support on $[0,1]$ and $f^{(m)} \varepsilon \mathscr{L}_p$, for one of the following cases:

$$m = 1, \qquad p = 2, \quad p = \infty$$

$$m = 2, \quad 1 \leq p \leq 2, \quad p = \infty$$

$$m = 3, \quad 1 \leq p \leq 2, \quad p = \infty$$

Let F_n be the sample c.d.f. based on n independent observations from F, and let $\hat{F}_n(x)$ be the cubic spline of interpolation to F_n at the points jh, $j = 0, 1, \ldots, \ell+1$; $(\ell+1)h = 1$, which satisfies the boundary conditions $\hat{F}_n'(0) = \hat{a}_1$, $\hat{F}_n'(1) = \hat{b}_1$, where $\hat{a}_1$ and $\hat{b}_1$ are given by (2.12) and (2.13). Let $\hat{f}_n(x) = \frac{d}{dx}\hat{F}_n(x)$, and suppose h is chosen as $h = k_n/n$,

$$k_n = \left[\frac{1}{(2m-2/p)} \frac{B}{||f^{(m)}||_p^2 A}\right]^{1/(2m+1-2/p)} n^{-(2m-2/p)/(2m+1-2/p)}$$

where A and B are given by (3.16) and (3.17). Then

$$E[f(x) - \hat{f}_n(x)]^2 \leq D\, n^{-(2m-2/p)/(2m+1-2/p)} \tag{3.18}$$

with

$$D = \frac{(2m+1-2/p)}{(2m-2/p)^{(2m-2/p)}} \left(||f^{(m)}||_p^2\, AB^{2m-2/p}\right)^{1/(2m+1-2/p)} . \tag{3.19}$$

APPENDIX

This appendix contains the proofs of Lemmas 2-6. The proofs are carried out where the knots $\{s_i\}_{i=1}^{\ell}$ do not necessarily satisfy $s_{i+1} - s_i = h$, but only $0 = s_0 < s_1 < \ldots < s_\ell < s_{\ell+1} = 1$. The purpose of this generality is to allow the lemmas to be referenced for a later report which deals with the situation where the knots are determined by the order statistics. Let I_j be the interval $[s_j, s_{j+1}]$, for $j = 0, 1, \ldots, \ell$.

<u>Lemma 2.</u>

Let $\tilde{Q}'_x$ be the projection of Q'_x onto $\mathcal{S}(\bar{s})$, and let $x \varepsilon I_j$. Then

$$||Q'_x - \tilde{Q}'_x||_Q^2 \leq (1/3)(s_{j+1} - s_j), \quad j = 0, 1, \ldots, \ell.$$

<u>Proof</u>: For $x \varepsilon I_j$, define R'_x in $\mathcal{H}_Q$ by

$$R'_x = \frac{1}{(s_{j+1} - s_j)} (Q_{s_{j+1}} - Q_{s_j}) \; .$$

Since $R'_x \varepsilon \mathcal{S}_\ell(\bar{s})$ and $\tilde{Q}'_x$ is the projection of Q'_x onto $\mathcal{S}_\ell(\bar{s})$

$$||Q'_x - \tilde{Q}'_x||_Q \leq ||Q'_x - R'_x||_Q \; . \qquad \text{(A1.1)}$$

To compute the square of the right side of (A1.1), note from (2.4) that

$$Q'_x(0) = 0$$

$$\frac{d}{ds} Q'_x(s)\Big|_{s=0} = 1$$

$$\frac{d^2}{ds^2} Q'_x(s) = 1 \quad s < x$$

$$= 0 \quad s > x .$$

After some calculations,

$$R'_x(0) = 0$$

$$\frac{d}{ds} R'_x(s)\Big|_{s=0} = 1$$

$$\frac{d^2}{ds^2} R'_x(s) = 1 \quad , 0 \leq s \leq s_j$$

$$= \frac{(s_{j+1} - s)}{(s_{j+1} - s_j)} , \; s_j \leq s \leq s_{j+1}$$

$$= 0 \quad , s_{j+1} \leq s \leq 1 .$$

$$\frac{d^2}{ds^2} Q'_x(s) - \frac{d^2}{ds^2} R'_x(s) = 0, \quad \text{for} \quad s \notin I_j$$

and

$$||Q'_x - R'_x||_Q^2 = \frac{1}{(s_{j+1} - s_j)^2} \left[\int_{s_j}^{x} (u-s_j)^2 du + \int_{x}^{s_{j+1}} (s_{j+1} - u)^2 du \right]^2$$

$$\leq \frac{1}{3}(s_{j+1} - s_j) .$$

Lemma 3.

Let $F \varepsilon \mathcal{H}_Q$ satisfy $F^{(iv)} = \rho \varepsilon \mathcal{L}_2[0,1]$. Let $\tilde{F}$ be the projection of F onto $\mathcal{S}_\ell(\bar{s})$. Then,

$$||F-\tilde{F}||_Q^2 \leq \sum_{j=0}^{\ell} (s_{j+1} - s_j)^{5-2/p} \left[\int_{s_j}^{s_{j+1}} |\rho(\xi)|^p \, d\xi \, . \right]^{2/p} \tag{A2.1}$$

If $s_{j+1} - s_j = h$, $j = 0, 1, \ldots, \ell$, and $1 \leq p \leq 2$,

$$||F-\tilde{F}||_Q^2 \leq h^{5-2p} \, ||F^{(iv)}||_p^2 \, .$$

Proof: First we show that $F^{(iv)} = \rho$ implies that

$$F(t) = \int_0^1 Q(t,s)\rho(s)ds + c_1Q_0(t) + c_2Q_1(t) + c_3Q_0'(t) + c_4Q_1'(t) \, . \tag{A2.2}$$

for some $\{c_i\}$. But $Q_0(t,s) = \int_0^1 (s-u)_+ \, (t-u)_+ \, du$ is the Green's function for the operator D^4, with boundary conditions

$$G^{(\nu)}(0) = 0, \quad \nu = 0, 1$$

$$G^{(\nu)}(1) = 0, \quad \nu = 2, 3 \, .$$

Thus, F always has a representation

$$F(t) = \int_0^1 Q_0(t,s)\rho(s)ds + \sum_{i=0}^{3} d_i t^i \, . \tag{A2.3}$$

But

$$\int_0^1 Q_0(t,s)\rho(s)ds = \int_0^1 Q(t,s)\rho(s)ds - (1+st) .$$

Since $Q_0(t)$, $Q_0'(t)$, $Q_1(t)$ and $Q_1'(t)$ span the same space as $\{1, t, t^2, t^3\}$ $\{c_i\}$ can always be found so that (A2.2) equals (A2.3).

Next, if v is any element in $\mathcal{H}_Q$ of the form

$$v = \sum_{i=0}^{\ell+1} c_i Q_{s_i} + aQ_0' + bQ_1'$$

Then, since $v \varepsilon \mathcal{S}_\ell(\bar{s})$,

$$||F-\tilde{F}||_Q \leq ||F-v||_Q . \qquad \text{(A2.4)}$$

The proof now proceeds by finding an element $v \varepsilon \mathcal{S}_\ell(\bar{s})$ so that the right hand side of (A2.4) is bounded by the right hand side of (A2.1). For $x \varepsilon I_j$, define $R_x \varepsilon \mathcal{S}_\ell(\bar{s})$ by

$$R_x = \frac{(s_{j+1} - x)}{(s_{j+1} - s_j)} Q_{s_j} + \frac{(x - s_j)}{(s_{j+1} - s_j)} Q_{s_{j+1}}, \qquad j = 0, 1, \ldots, \ell .$$

Define $v \varepsilon \mathcal{S}_\ell(\bar{s})$ by

$$v = \int_0^1 R_x \rho(x)dx + c_1 Q_0 + c_2 Q_0' + c_3 Q_1 + c_4 Q_1'$$

$$\equiv \sum_{j=0}^{\ell} \left\{ Q_{s_j} \int_{s_j}^{s_{j+1}} \frac{(s_{j+1} - x)}{(s_{j+1} - s_j)} \rho(x)dx + Q_{s_{j+1}} \int_{s_j}^{s_{j+1}} \frac{(x - s_j)}{(s_{j+1} - s_j)} \rho(x)dx \right\}$$

$$+ c_1 Q_0 + c_2 Q_0' + c_3 Q_1 + c_4 Q_1' .$$

Now

$$F-v = \int_0^1 (Q_x - R_x)\rho(x)dx$$

and, by the properties of the reproducing kernel, it can be shown that

$$||F-v||_Q^2 = \int_0^1 \int_0^1 \rho(x)\rho(x') < Q_x-R_x, Q_{x'}-R_{x'} > \ dxdx' \ .$$

Since

$$Q_x(0) - R_x(0) = 0$$

$$\frac{d}{ds}(Q_x(s) - R_x(s))\Big|_{s=0} = 0$$

$$\frac{d^2}{ds^2}(Q_x(s) - R_x(s)) = (x-s)_+ - \frac{(x-s_j)(s_{j+1}-s)}{(s_{j+1}-s_j)}, \quad x\varepsilon I_j, \ s\varepsilon I_j$$

$$= 0, \quad x\varepsilon I_j, \quad s\varepsilon I_k, \quad k \neq j \ .$$

it follows that

$$< Q_x-R_x, Q_{x'}-R_{x'} > = 0$$

if $x\varepsilon I_j$, $x'\varepsilon I_k$ with $j \neq k$. Thus

$$||F-v||_Q^2 \leq \sum_{j=0}^{\ell} \left\{ \int_{s_j}^{s_{j+1}} |\rho(x)| \ ||Q_x-R_x||_Q dx \right\}^2 .$$

Furthermore

$$||Q_x - R_x||_Q^2 = \int_{s_j}^{s_{j+1}} \left[(x-s)_+ - \frac{(x - s_j)_+}{(s_{j+1} - s_j)} (s_{j+1} - s)_+\right]^2 ds$$

$$\leq (s_{j+1} - s_j)^3, \quad x \varepsilon I_j,$$

so that

$$||F-v||_Q^2 \leq \sum_{j=0}^{\ell} (s_{j+1} - s_j)^3 \left[\int_{s_j}^{s_{j+1}} |\rho(t)|dt\right]^2 .$$

For $\frac{1}{p} + \frac{1}{p'} = 1$, a Hölder inequality gives

$$\int_{I_j} |\rho(t)|dt \leq \left[\int_{I_j} dt\right]^{1/p'} \left[\int_{I_j} |\rho(t)|^p dt\right]^{1/p}$$

$$= (s_{j+1} - s_j)^{1-1/p} \left[\int_{I_j} |\rho(t)|^p dt\right]^{1/p} .$$

Thus,

$$||F-v||_Q^2 \leq \sum_{j=0}^{\ell} (s_{j+1} - s_j)^{5-2/p} \left[\int_{I_j} |\rho(t)|^p dt\right]^{2/p}.$$

If $(s_{j+1} - s_j) = h$ and $1 \leq p \leq 2$, then a Hölder inequality gives

$$||F-v||_Q^2 \leq h^{5-2/p} \left[\int_0^1 |\rho(t)|^p dt\right]^{2/p} .$$

Lemma 4.

Let $F \varepsilon \mathcal{H}_Q$ satisfy $F^{(iii)} = \eta \varepsilon \mathcal{L}_2[0,1]$. Let $\tilde{F}$ be the projection of F onto $\mathcal{S}_\ell(\bar{s})$. Then

$$||F-\tilde{F}||_Q^2 \leq \frac{1}{3} \sum_{j=0}^{\ell} (s_{j+1} - s_j)^{3-2/p} \left[\int_{s_j}^{s_{j+1}} |\eta(\xi)|^p d\xi \right]^{2/p}$$

If $(s_{j+1} - s_j) = h$, and $1 \leq p \leq 2$, then

$$||F-\tilde{F}||_Q^2 \leq \frac{1}{3} h^{3-2/p} \, ||F^{(iii)}||_p^2 \; .$$

Proof: As in the proof of Lemma 3, by the Green's function properties of $Q_x'(s)$, there exist c_1, c_2, c_3, c_4 such that

$$F(t) = \int_0^1 Q_x'(t)\eta(x)ds + c_1 Q_0(t) + c_2 Q_0'(t) + c_3 Q_1(t) + c_4 Q_1'(t) \; .$$

Let

$$v = \int_0^1 R_x' \; \eta(x)dx + c_1 Q_0 + c_2 Q_0' + c_3 Q_1 + c_4 Q_1' \; .$$

where R_x' is defined as in the proof of Lemma 2,

$$R_x' = \frac{1}{(s_{j+1} - s_j)} (Q_{s_{j+1}} - Q_{s_j}) \quad \text{for} \quad x \varepsilon I_j \; .$$

Then

$$||F-\tilde{F}||_Q^2 \leq ||F-v||_Q^2 = \int_0^1 \int_0^1 \eta(x)\eta(x') < Q_x'-R_x', Q_{x'}'-R_{x'}' > dxdx' \; .$$

Also, it can be shown that

$$< Q'_x - R'_x, Q'_{x'} - R'_{x'} > = 0 \quad \text{if} \quad x \varepsilon I_j, \; x' \varepsilon I_k, \; j \neq k;$$

so that

$$||F-v||_Q^2 \leq \sum_{j=0}^{\ell} \left\{ \int_{s_j}^{s_{j+1}} |\eta(x)| \; ||Q'_x - R'_x|| dx \right\}^2 .$$

By Lemma 2,

$$||Q'_x - R'_x||^2 \leq (1/3)(s_{j+1} - s_j) \quad \text{for} \quad x \varepsilon I_j$$

from which the result follows as in Lemma 3.

<u>Lemma 5.</u>

Let $S(x) = S(x, \bar{s}; \bar{\varepsilon}_a, \bar{\varepsilon}, \bar{\varepsilon}_b)$ be the cubic spline of interpolation defined by (2.8) with

$$\bar{s} = (s_1, s_2, \ldots, s_\ell)$$

$$\bar{\varepsilon}_a = (\varepsilon'_0, \varepsilon_0) \quad , \quad \varepsilon_0 = 0$$

$$\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell)$$

$$\bar{\varepsilon}_b = (\varepsilon_{\ell+1}, \varepsilon'_{\ell+1}), \quad \varepsilon_{\ell+1} = 0 .$$

Let

$$\Delta_i = (s_{i+1} - s_i), \quad i = 0, 1, \ldots, \ell,$$

$$\psi_i = (\varepsilon_{i+1} - \varepsilon_i), \quad i = 0, 1, \ldots, \ell .$$

Then, for $x \varepsilon I_j$,

$$\left|\frac{d}{dx} S(x)\right| \leq 8 \sum_{i=0}^{\ell} c_i \frac{\Delta_j}{\Delta_i} \frac{|\psi_i|}{\Delta_i} + \frac{8}{2^{j+1}} \frac{\Delta_j}{\Delta_0} |\varepsilon_0'| + \frac{8}{2^{\ell+2-j}} \frac{\Delta_j}{\Delta_\ell} |\varepsilon_{\ell+1}'| \qquad \text{(A4.1)}$$

where

$$c_i = \frac{1}{2^{|i-j|+1}} + \frac{1}{2^{|i+1\,-\,j|+1}} \qquad i = 0, 1, \ldots, \ell, \quad i \neq j$$

$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{8}, \qquad i = j \cdot$$

<u>Proof:</u>

Define

$$R_x' = \frac{1}{\Delta_j} (Q_{s_{j+1}} - Q_{s_j}) \quad \text{for} \quad x \varepsilon I_j \cdot$$

Then

$$\frac{d}{dx} S(x) = < S, Q'_x >_Q = < S, R'_x >_Q + < S, Q'_x - R'_x > .$$

Since $S(s_j) = \varepsilon_j$, $j = 0, 1, \ldots, \ell+1$,

$$< S, R'_x > = \frac{1}{\Delta_j} (\varepsilon_{j+1} - \varepsilon_j) = \frac{\psi_j}{\Delta_j} . \qquad \text{(A4.2)}$$

To study

$$< S, Q'_x - R'_x >$$

we note that

$$Q'_x(0) - R'_x(0) = 0$$

$$\frac{d}{ds} (Q'_x(s) - R'_x(s))\Big|_{s=0} = 0$$

$$\frac{d^2}{ds^2} [Q'_x(s) - R'_x(s)] = 0 \qquad , \quad s \notin I_j$$

$$= \frac{1}{\Delta_j} (s - s_j) \quad , \quad s_j \le s \le x$$

$$= \frac{1}{\Delta_j} (s_{j+1} - s), \quad x \le s \le s_j .$$

Since the spline S is a cubic between the knots $0, s_1, \dots, s_\ell, 1$, $\frac{d^2}{dx^2} S(x)$ is linear between the knots, and by the properties of cubic splines, continuous. Define κ_i by

$$\frac{d^2}{ds^2} S(s)\Big|_{s=s_i} = \kappa_i .$$

Thus

$$\frac{d^2}{ds^2} S(s) = \frac{1}{\Delta_j} [\kappa_j(s_{j+1} - s) + \kappa_{j+1}(s - s_j)] \quad \text{for} \quad s\varepsilon I_j,$$

and

$$< S, Q'_x - R'_x >_Q = \frac{1}{\Delta_j^2} \int_{s_j}^{x} (s-s_j)[\kappa_j(s_{j+1} - s) + \kappa_{j+1}(s-s_j)]ds$$

$$+ \frac{1}{\Delta_j^2} \int_{x}^{s_{j+1}} (s_{j+1} - s)[\kappa_j(s_{j+1} - s) + \kappa_{j+1}(s-s_j)]ds, \quad \text{for} \quad x\varepsilon I_j,$$

giving

$$|< S, Q'_x - R'_x >_Q| \le \Delta_j \max(|\kappa_j|, |\kappa_{j+1}|), \quad x\varepsilon I_j . \qquad \text{(A4.3)}$$

To proceed, we need to know the relationship between the κ_i and the data $\bar{s}, \bar{\varepsilon}_a, \bar{\varepsilon}, \bar{\varepsilon}_b$. By using a formula found in Kershaw, [9], equation (5), we may express this relationship for cubic splines. It is

$$(\kappa_0, \kappa_1, \ldots, \kappa_\ell, \kappa_{\ell+1}) = 6\, A^{-1}(\xi_0, \xi_1, \ldots, \xi_\ell, \xi_{\ell+1})$$

where A is the $(\ell+2) \times (\ell+2)$ matrix given by

$$A = \begin{bmatrix} 2 & 1 & & & & \\ \alpha_1 & 2 & 1-\alpha_1 & & 0 & \\ & \alpha_2 & 2 & 1-\alpha_2 & & \\ & & & & & \\ 0 & & & \alpha_\ell & 2 & 1-\alpha_\ell \\ & & & & 1 & 2 \end{bmatrix}$$

where

$$\alpha_i = \frac{\Delta_{i-1}}{\Delta_i + \Delta_{i-1}} \qquad i = 1, 2, \ldots, \ell$$

and

$$\xi_0 = (\psi_0 - \Delta_0 \varepsilon_0')/\Delta_0^2$$

$$\xi_i = \left(\frac{\psi_i}{\Delta_i} - \frac{\psi_{i-1}}{\Delta_{i-1}}\right) \Big/ \left(\Delta_i + \Delta_{i-1}\right), \qquad i = 1, 2, \ldots, \ell$$

$$\xi_{\ell+1} = -(\psi_\ell - \Delta_\ell \varepsilon_{\ell+1}')/\Delta_\ell^2 \,.$$

For $i = 1, 2, \ldots, \ell$, ξ_i is the second divided difference of $S(x)$ at (s_{i-1}, s_i, s_{i+1}).

We are now going to appeal to another result of Kershaw's, which gives bounds on the entries of A^{-1}. Let a^{rs}, $r, s = 0, 1, \ldots, \ell+1$, be the r, sth entry of A^{-1}. According to [8],

$$|a^{rs}| \leq \frac{4}{3} \frac{1}{2^{|r-s|+1}}, \quad r, s = 0, 1, \ldots, \ell+1 .$$

Therefore, since $\kappa_j = 6 \sum_{i=0}^{\ell+1} a^{ji} \xi_i$,

$$|\kappa_j| \leq 6 \cdot \frac{4}{3} \cdot \sum_{i=0}^{\ell+1} \frac{1}{2^{|i-j|+1}} |\xi_i| \tag{A4.4}$$

and, combining (A4.2), (A4.3), and (A4.4) gives

$$|< S, Q'_x - R'_x >| \leq \Delta_j \cdot 6 \cdot \frac{4}{3} \sum_{i=0}^{\ell+1} \frac{1}{2^{|i-j|+1}} |\xi_i|, \quad x \varepsilon I_j$$

and

$$\begin{aligned} |< S,Q'_x >| &\leq \frac{|\psi_j|}{\Delta_j} + 8 \cdot \sum_{i=1}^{\ell} \frac{1}{2^{|i-j|+1}} \left\{ \frac{|\psi_i|}{\Delta_i} + \frac{|\psi_{i-1}|}{\Delta_{i-1}} \right\} \frac{\Delta_j}{\Delta_i + \Delta_{i-1}} \\ &\quad + 8 \frac{1}{2^{j+1}} \frac{\Delta_j}{\Delta_0} \left\{ \frac{|\psi_0|}{\Delta_0} + |\varepsilon'_0| \right\} \\ &\quad + 8 \frac{1}{2^{\ell+2-j}} \frac{\Delta_j}{\Delta_\ell} \left\{ \frac{|\psi_\ell|}{\Delta_\ell} + |\varepsilon'_{\ell+1}| \right\} \\ &\leq 8 \sum_{i=0}^{\ell} c_i \frac{|\psi_i|}{\Delta_i} \frac{\Delta_j}{\Delta_i} + \frac{8}{2^{j+1}} \frac{\Delta_j}{\Delta_0} |\varepsilon'_0| + \frac{8}{2^{\ell+2-j}} \frac{\Delta_j}{\Delta_\ell} |\varepsilon'_{\ell+1}| \end{aligned}$$

where

$$c_i = \frac{1}{2^{|i-j|+1}} + \frac{1}{2^{|i+1-j|+1}}, \quad i = 0, 1, 2, \ldots, \ell, \quad i \neq j$$

$$c_j = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{8}.$$

We remark that the lack of a generalization of this lemma for higher degree splines is the stumbling block in generalizing the main theorem to higher m.

Lemma 5.

Suppose $f^{(m)} \varepsilon \mathcal{L}_p$ on $[0,1]$. Let F_n be the sample c.d.f. for n independent observations from f. Let

$$\hat{f}_n(0) = \frac{d}{dx} \sum_{\nu=0}^{m} \ell_{0,\nu}(x)\Big|_{x=0} F_n(\nu h)$$

$$\hat{f}_n(1) = \frac{d}{dx} \sum_{\nu=0}^{m} \ell_{1,\nu}(x)\Big|_{x=1} F_n(1-\nu h)$$

where $\ell_{0,\nu}$ and $\ell_{1,\nu}$ are the Lagrange polynomials defined in (2.10) and (2.11). (For $m = 1$, $\hat{f}_n(0) = \frac{1}{n} F_n(h)$.) Then

$$\left.\begin{array}{l} E(f(0) - \hat{f}_n(0))^2 \\ \\ E(f(1) - \hat{f}_n(1))^2 \end{array}\right\} \leq \frac{8m^{2m-2/p}}{[(m-1)!]^2} ||f^{(m)}||_p^2 \, h^{2m-2/p} + 2m^3(m+1)^2 \frac{\Lambda}{nh}. \qquad \text{(A5.1)}$$

Proof:

$$\left|f(0) - \hat{f}_n(0)\right| \leq \left|f(0) - \frac{d}{dx} \sum_{\nu=0}^{m} \ell_{0,\nu}(x)\Big|_{x=0} F(\nu h)\right| + \left|\frac{d}{dx} \sum_{\nu=0}^{m} \ell_{0,\nu}(x)[F(\nu h) - F_n(\nu h)]\right|$$

(A5.2)

By combining Lemma 3.1 of [21], and Theorem 3 of [20], and noting that $\prod_{j=0}^{m} (0-jh) = 0$ in equation (3.28) of [20], it can be shown that

$$\left|f(0) - \frac{d}{dx} \sum_{\nu=0}^{m} \ell_{0,\nu}(x)\Big|_{x=0} F(\nu h)\right|^2 \leq \left[\frac{2}{(m-1)!}\right]^2 \left[\int_0^{mh} |f^{(m)}(\xi)|^p d\xi\right]^{2/p} (mh)^{m-1/p}, \quad \text{(A5.3)}$$

$$p \geq 1, \; m = 1, 2, \ldots$$

It can be verified, for $m = 1, 2, 3$, that

$$\left| \frac{d}{dx} \ell_{0,\nu}(x)\Big|_{x=0} \right| \leq \frac{m}{h}. \quad \text{(A5.4)}$$

Now

$$F_n(\nu h) = \frac{\#\text{ observations in } [0,\nu h]}{n},$$

and hence $nF_n(\nu h)$ is binomial $B(n, \sum_{j=1}^{\nu} p_j)$, where $p_j = \int_{(j-1)h}^{jh} f(\xi)d\xi$ and hence

$$E[F(\nu h) - F_n(\nu h)]^2 = \left(\sum_{j=1}^{\nu} p_j\right)(1 - \sum_{j=1}^{\nu} p_j)/n \leq \frac{\Lambda mh}{n}. \quad \text{(A5.5)}$$

Putting together (A5.2) (A5.3) (A5.4) and (A5.5) gives

$$E(f(0) - \hat{f}_n(0))^2 \leq \frac{8m^{2m-2/p}}{((m-1)!)^2} ||f^{(m)}||_p^2 \, h^{2m-2/p}$$

$$+ 2m^3(m+1)^2 \frac{\Lambda}{nh}.$$

The proof is carried out similarly for $x = 1$.

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