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SOME EXPONENTIALLY DECREASING ERROR BOUNDS FOR A NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

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ABSTRACT

Convergence properties of a class of least-squares methods for finding approximate inverses of the Laplace transform are obtained using reproducing kernel Hilbert space techniques.

SOME EXPONENTIALLY DECREASING ERROR BOUNDS FOR A NUMERICAL INVERSION OF THE LAPLACE TRANSFORM

M. Z. Nashed and Grace Wahba[1]

1. Introduction and Preliminaries

We obtain error bounds for certain approximations to the inverse Laplace transform. Suppose

$$\int_0^\infty e^{-st} f(t)dt = F(s) . \tag{1.1}$$

We wish to construct an approximation $f_n(t)$ to the inverse transform $f(t)$, using $n+1$ values $F(s_i)$, $i = 0,1,\cdots,n$ of F. The problem of inversion of the Laplace transform, being an ill-posed problem, gives rise to many interesting and challenging numerical and analytic investigations. The monographs of Bellman, Kalaba, and Lockett [2] and Krylov and Skoblya [4] are devoted to this important problem, where a number of methods are developed. A synopsis of the difficulties and the rationale of various approaches to the numerical inversion of the Laplace transform are given in Bellman [1, Chapter 19]. In the present note we consider only a very simple method used in [2, Chapter 2] and more recently by Schoenberg [7].

We suppose that $f \in L_2(\alpha)$, where $L_2(\alpha)$ is the Hilbert space of real-valued functions on $[0,\infty)$, square integrable with respect to the weight function $w_\alpha(t) = e^{2\alpha t}$

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(α is a fixed constant). Denote the inner product and norm in $L_2(\alpha)$ by

$$(f,g)_\alpha = \int_0^\infty f(t)g(t)e^{2\alpha t}dt \quad , \quad \|f\|_\alpha = (f,f)_\alpha^{\frac{1}{2}} .$$

Let s_i, $i = 0,1,\cdots,n$ be $n+1$ distinct points in $[0,\infty)$. Let the approximate solution f_n to (1.1) be the solution to the minimization problem:

Find $f \in L_2(\alpha)$ to minimize $\|f\|_\alpha$ subject to

$$\int_0^\infty e^{-s_i t} f(t)dt = F(s_i), \; i = 0,1,\cdots,n . \tag{1.2}$$

Let the functions ψ_{s_i} be defined by

$$\psi_{s_i}(t) = e^{-(s_i + 2\alpha)t} , \; i = 0,1,\cdots,n .$$

Provided $s_0 + \alpha > 0$, $\psi_{s_i} \in L_2(\alpha)$ and the conditions (1.2) may be rewritten

$$(\psi_{s_i} , f)_\alpha = F(s_i), \; i = 0,1,\cdots,n . \tag{1.3}$$

Let Q_n be the Gram matrix of ψ_{s_i} , $i = 0,1,\cdots,n$. The ij^{th} entry of Q_n is given by

$$(\psi_{s_i} , \psi_{s_j})_\alpha = \int_0^\infty e^{-(s_i + s_j + 2\alpha)t} dt = (s_i + s_j + 2\alpha)^{-1} .$$

Thus Q_n is a generalization of a section of a Hilbert matrix, and hence Q_n is nonsingular (see Isaacson and Keller [3, p. 217]). It is easy to show, using (1.3), that the solution

f_n to the minimization problem is in the span of ψ_i, $i = 0,1,\cdots,n$, and is given by

$$f_n = (\psi_{s_0}, \psi_{s_1}, \cdots, \psi_{s_n}) Q_n^{-1} (F(s_0), F(s_1), \cdots, F(s_n))' . \qquad (1.4)$$

The ijth entry q^{ij} of Q_n^{-1} is given by the formula:

$$q^{ij} = (s_i + s_j + 2\alpha) A_j(-(s_i + \alpha)) A_j(-(s_j + \alpha)),$$
$$i,j = 0,1,\cdots,n,$$

where

$$A_i(x) = \prod_{k \neq i} \frac{s_k + \alpha - x}{s_k - s_i}$$

(see Isaacson and Keller [3, p. 218]).

Schoenberg [7] discusses the case $\alpha = -\frac{1}{2}$, $s_j = j + 1$, $j = 0,1,\cdots,n$. He gives the solution to the minimization problem (1.2) in the form

$$f_n(t) = S_n(e^{-t})$$

where $S_n(x) = \sum_{v=0}^{n} c_v P_v(2x - 1)$, $P_v(x)$ being the classical Legendre polynomials, and

$$c_v = (2v + 1) \sum_{i=0}^{v} (-1)^{v+i} \binom{v+i}{v} \binom{v}{i} F(i + 1) .$$

2. The Main Result

We now give some $L_2(\alpha)$-convergence properties of this method and error bounds for $\alpha > 0$, $\alpha(s_{j+1} - s_j)^{-1}$ and $n(s_{j+1} - s_j)$ large.

Theorem. <u>Let</u> f_n <u>be given by</u> (1.4), <u>where</u> $\alpha > 0$, $s_j = \frac{j}{n} T$, $j = 0,1,\cdots,n$, <u>with</u> T <u>a positive number no less than</u> 2α. <u>Suppose</u> $f \in L_2(\alpha)$, <u>and furthermore, has a representation of the form</u>

$$f(t) = e^{-2\alpha t} \int_0^\infty e^{-ts} \rho(s)\,ds \tag{2.1}$$

<u>where</u> $\int_0^\infty |\rho(s)|\,ds < \infty$. <u>Then</u>

$$\int_0^\infty [f(t) - f_n(t)]^2 e^{2\alpha t}\,dt \le$$

$$2\{\frac{e^4}{2\alpha\pi^{\frac{1}{2}}}(\frac{n\alpha}{T})^{\frac{3}{2}} e^{-2n\alpha/T}(1+O(\frac{T}{n\alpha})) + \int_T^\infty\int_T^\infty \frac{\rho(s)\rho(t)}{s+t+2\alpha}\,dsdt\ . \tag{2.2}$$

Proof. Let K be the operator which maps $f \in L_2(\alpha)$ into its Laplace transform:

$$(Kf)(s) = \int_0^\infty e^{-st} f(t)\,dt \equiv F(s)\ , \qquad s \ge 0\ .$$

Using properties of reproducing kernel Hilbert spaces (RKHS), for more details see e.g. [6],[8], $K(L_2(\alpha))$ is the RKHS of real-valued functions on $[0,\infty)$ with the reproducing kernel $Q(s,t)$ given by

$$Q(s,t) = (\psi_s, \psi_t)_\alpha = (s+t+2\alpha)^{-1}, \qquad 0 \le s,t < \infty .$$

The condition (2.1) is equivalent to

$$F(s) = \int_0^\infty Q(s,t)\rho(t)dt \equiv \int_0^\infty e^{-st}f(t)dt . \qquad (2.3)$$

Denote by $Q_x(s)$ the real-valued function of s on $[0,\infty)$ defined by $Q_x(s) = Q(x,s)$. Thus Q_x is the representer of the evaluation functional at x in H_Q. Let

$$F_n(s) = (Q_{s_1}(s), Q_{s_2}(s), \cdots, Q_{s_n}(s))Q_n^{-1}(F(s_1), F(s_2), \cdots, F(s_n))' .$$

Since $Q_{s_i} = K\psi_{s_i}$, $F_n = Kf_n$ and, furthermore, F_n is the orthogonal projection in H_Q of F onto the subspace of H_Q spanned by the functions Q_{s_i}, $i = 0,1,\cdots,n$ (Q_n is the Gram matrix of $Q_{s_0},\cdots Q_{s_n}$ in H_Q). By the properties of RKHS, and the fact that

$$f \in L_2(\alpha) \text{ and } Kf = 0 \Rightarrow f = 0 ,$$

there is an isometric isomorphism between $L_2(\alpha)$ and H_Q whereby

$$F \in H_Q \sim f \in L_2(\alpha) \Leftrightarrow F = Kf .$$

Thus

$$||F - F_n||_Q = ||f - f_n||_\alpha ,$$

where $||\cdot||_Q$ is the norm in H_Q. Thus, the proof will be effected if we show that $||F - F_n||_Q^2$ is bounded by the right hand side of (2.2).

Now, it can be shown, either directly, or using the properties of RKHS,[+] that

$$||F - F_n||_Q^2 = \int_0^\infty \int_0^\infty \rho(s)\rho(t)[Q(s,t) - Q_n(s,t)]dsdt$$

where

$$Q_n(s,t) = (Q_{s_0}(s),\cdots Q_{s_n}(s))Q_n^{-1}(Q_{s_0}(t),\cdots,Q_{s_n}(t)) ,$$

and $Q_n(s,t)$ and $E_n(s,t)$ defined by

$$E_n(s,t) = Q(s,t) - Q_n(s,t)$$

are positive definite kernels.

Thus, we may write

$$||F-F_n||_Q^2 = \int_0^\infty \int_0^\infty \rho(s)\rho(t)E_n(s,t)dsdt \leq 2\{\int_0^T \int_0^T \rho(s)\rho(t)E_n(s,t)dsdt + \int_T^\infty \int_T^\infty \rho(s)\rho(t)E_n(s,t)dsdt\}$$

$$\leq 2\{\int_0^T \int_0^T \rho(s)\rho(t)E_n(s,t)dsdt + \int_T^\infty \int_T^\infty Q(s,t)\rho(s)\rho(t)dsdt\} .$$

[+]Recall that $\langle Q_s, Q_t \rangle_Q = Q(s,t)$

$$\leq 2\{\sup_{0\leq s\leq T} E_n(s,s)\left[\int_0^T |\rho(s)|ds\right]^2$$

$$+ 2\int_T^\infty\int_T^\infty \frac{\rho(s)\rho(t)}{(s+t+2\alpha)}\,dsdt\} ,$$

and, it remains to find a bound on $\sup_{0\leq s\leq T} E_n(s,s)$. This is done as follows:

Note that

$$Q(s,t) = \int_0^\infty G(s,u)G(t,u)du$$

where

$$G(s,u) = e^{-(\alpha+s)u} , \qquad s,u \geq 0$$

and furthermore

$$E_n(s,s) = \inf\{\int_0^\infty (G(s,u)-\sum_{i=0}^n c_i G(s_i,u))^2 du : c_i \in R,\ i = 0,\cdots,n\} ,$$

so that

$$E_n(s,s) \leq \int_0^\infty (G(s,u)-\sum_{i=0}^n c_i G(s_i,u))^2 du , \tag{2.4}$$

for any real $c_0,c_1,\cdots,c_n$.

Let s be fixed, with $s_j \equiv j\,T/n \leq s < s_{j+1} \equiv (j+1)\,T/n$, and suppose $j \leq n-(N-1)$, where $N-1$ is the greatest integer in $\alpha n/T$.

Let $G_u(s)$ be the function of s given by

$$G_u(s) = G(s,u) \equiv e^{-(\alpha+s)u} , \qquad s,u \geq 0 ,$$

and let $c_i = c_i(s)$, $i = 0,1,\cdots,n$, be defined by

$$\sum_{i=0}^{n} c_i G_u(s_i) = \sum_{i=0}^{N-1} \rho_{iN}(s) G_u(s_{j+i})$$

where $\rho_{iN}(s)$ is the polynomial of degree $N-1$, which takes on the value 1 at $s = s_{j+i}$ and the value 0 at $s = s_{j+k}$, $k = 0,1,\cdots,N-1$, $k \neq i$. Thus

$$\sum_{i=0}^{N-1} \rho_{iN}(s) G_u(s_{j+i})$$

is the Lagrange polynomial in s interpolating to $G_u(s)$ at the points $s_j, s_{j+1},\cdots,s_{j+N-1}$. By the Newton form of the remainder for Lagrange interpolation,

$$G_u(s) - \sum_{i=0}^{N-1} \rho_{iN}(s) G_u(s_{j+i}) = \prod_{i=0}^{N-1} (s-s_{j+i}) G_u[s_j, s_{j+1},\cdots,s_{j+N-1}, s], \quad (2.5)$$

where $G_u[s_j, s_{j+1},\cdots,s_{j+N-1}, s]$ is the N^{th} divided difference of $G_u(x)$ at the points $x = s_j,\cdots,s_{j+N-1}, s$. Thus, there exists some $\theta \varepsilon [s_j, s_{j+N-1}]$ such that

$$\begin{aligned} G_u[s_j, s_{j+1},\cdots,s_j+N-1, s] &= \frac{1}{N!} \frac{\partial^N}{\partial x^N} G_u(x) \Big|_{x=\theta} \\ &= \frac{u^N}{N!} e^{-(\theta+\alpha)u} \end{aligned} \qquad (2.6)$$

Substituting (2.6) into (2.5), and (2.5) into (2.4) gives

$$E_n(s,s) \leq \left| \prod_{i=0}^{N-1} (s-s_{j+i}) \right|^2 \int_0^\infty \frac{u^{2N}}{(N!)^2} e^{-2(\theta+\alpha)u} \, du$$

$$= \left| \prod_{i=0}^{N-1} (s-s_{j+i}) \right|^2 \frac{(2N)!}{(N!)^2} [2^{2N+1}(\theta+\alpha)^{2N+1}]^{-1}$$

$$\leq \frac{1}{2\alpha} \frac{(2N)!}{(N!)^2 2^{2N}} \prod_{i=1}^{N-1} \left[\frac{s_{j+i}-s_j}{\alpha}\right]^2 , \quad \text{for } s \in [s_j, s_{j+i}) . \qquad (2.7)$$

Now, use $s_{j+i}-s_j = iT/n$, $N-1 \leq \alpha n/T < N$ to obtain

$$\prod_{i=1}^{N-1} \frac{(s_{j+i}-s_j)}{\alpha} \leq \prod_{i=1}^{N-1} \frac{i}{N-1} .$$

Furthermore,

$$\log \prod_{i=1}^{N-1} \frac{i}{N-1} = \sum_{i=1}^{N-1} \log \left(\frac{i}{N-1}\right) \leq (N-1) \int_{\frac{1}{N-1}}^{1} \log u \, du =$$

$$- (N-2) + \log (N-1) ,$$

hence

$$\prod_{i=1}^{N-1} \frac{i}{N-1} \leq (N-1)e^{-(N-2)} \leq e^2 \left(\frac{\alpha n}{T}\right) e^{-(\alpha n)/T} .$$

By Stirling's formula,

$$\frac{(2N)!}{(N!)^2 2^{2N}} = \frac{1}{\sqrt{\pi N}} \left(1 + O\left(\frac{1}{N}\right)\right) .$$

$$\leq \frac{1}{\sqrt{\pi}} \left(\frac{\alpha n}{T}\right)^{-\frac{1}{2}} \left(1 + O\left(\frac{T}{\alpha n}\right)\right) .$$

Thus, for $s < s_{n-(N-1)}$

$$E_n(s,s) \le \frac{1}{2\alpha} \frac{e^4}{\pi^{\frac{1}{2}}} \left(\frac{\alpha n}{T}\right)^{\frac{3}{2}} e^{-2\alpha n/T} \left(1 + O\left(\frac{T}{\alpha n}\right)\right) .$$

The same bound may be obtained for $s \ge s_{n-(N-1)}$ provided $n-(N-1) \ge N-1$, by approximating $G_u(s)$ in (2.6) by the $G_u(s_i)$ with s_i to the left of s. The condition $T \ge 2\alpha$ insures that $n-(N-1) \ge N-1$, and the theorem is proved. ∎

3. Extensions

When $\alpha \le 0$ a similar convergence theorem can be proved, if $s_j = s_0 + \frac{j}{n} T$, where $s_0 + \alpha > 0$. It is necessary to assume that $\int_0^{s_0} |\rho(s)| ds = 0$. Then (2.2) can be shown to hold with the right hand side of (2.2) having α replaced by $\alpha + s_0$, and the lower limits on the double integral $T + s_0$ instead of T. The left hand side has f replaced by $f^\dagger$, where $f^\dagger$ is that element in $L_2(\alpha)$ of minimal $L_2(\alpha)$-norm satisfying

$$\int_0^\infty e^{-st} f(t)dt = F(s), \qquad s \ge s_0 .$$

The modifications in the proof occur by noting the following facts which can be easily established:

(1) There is an isometric isomorphism between $L^\dagger(\alpha)$ and H_Ω where $L^\dagger(\alpha)$ is the quotient space $L(\alpha)/N(K)$, $N(K) = \{f \in L_2(\alpha), \int_0^\infty e^{st} f(t)dt = 0, s \ge s_0\}$,

and H_Q now has the reproducing kernel $Q(s,t)$, $s,t \geq s_0$.

(2) The condition $\int_0^{s_0} |\rho(s)| ds = 0$ insures that (2.3) holds.

(3) α is replaced by $s_0 + \alpha$ in (2.7) and the subsequent argument; and $N-1$ is the greatest integer in $(s_0 + \alpha)n/T$.

Finally we remark that the error bounds and convergence properties of the approximations to the inverse transform rely heavily on the particular kernel associated with the Laplace transform, and are not a special case of other results on regularization and approximation of ill-posed linear operator equations [5], using reproducing kernel space methods.

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